Limit Superior and Limit Inferior, I: Bounded Sequences

Note Title 6/9/2015

Let (a_n) denote a bounded sequence of real numbers, and consider the sequences

$$t_n := \inf(a_n, a_{n+1}, a_{n+2}, \ldots)$$

$$T_n := \sup(a_n, a_{n+1}, a_{n+2}, \ldots)$$

For example, suppose $a_n = (-1)^n$. Then $t_n = -1 \;\; \forall n$ and $T_n = +1 \;\; \forall n$.

Notice that a_n certainly does not converge in this case, but that (t_n) and (T_n) trivially converge.

We have:

$$\inf_k a_k = t_1 \leq t_n \leq T_n \leq T_1 = \sup_k a_k$$

We have two bounded, monotone sequences (t_n) and (T_n) and so both must converge.

In this case (i.e., for (a_n) bounded) we define limit superior as

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \overline{T_n}$$

We define limit inferior as

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} t_n .$$

For our example $(-1)^n$, we have:

$\liminf_{n\to\infty} (-1)^n = -1$, and $\limsup_{n\to\infty} (-1)^n = +1$.