Limit Superior and Limit Inferior, II: Unbounded Sequences

Note Title 6/10/2015

If the initial sequence (a_n) is unbounded, we can proceed similarly as before, except in this case the sequences (t_n) and (T_n) may not converge to a finite value.

For example, consider $a_n = \begin{cases} 0 & n \text{ odd} \\ n & n \text{ even} \end{cases}$.

$\Rightarrow T_n = \infty \quad \forall n$ and $t_n = 0 \quad \forall n$

Even though we might say something like

"$\overline{T}_n \to \infty$" as $n \to \infty$

we would not say $\overline{T}_n$ converges.

Noneless, $\inf\limits_{n} \overline{T}_n = \infty$.

In this way, it's natural for unbounded sequences to work with inf and sup.

We define the limit inferior of (a_n) as

$$\liminf_{n \to \infty} a_n = \sup_{n \geq 1} t_n$$

Likewise, we define the limit superior of (a_n) as

$$\limsup_{n \to \infty} a_n = \inf_{n \geq 1} T_n .$$

Examples

Return to our example $a_n = (-1)^n \ \forall n.$

Here, $t_n = -1 \ \forall n$ and $T_n = +1 \ \forall n$

$$\Rightarrow \quad \liminf_{n\to\infty} a_n = \sup_{n\geq 1} t_n = -1$$

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} T_n = +1.$$

For $a_n = \begin{cases} 0 & n \text{ odd} \\ n & n \text{ even} \end{cases}$

$$\liminf_{n\to\infty} a_n = \sup_{n\geq 1} t_n = 0$$

$$\limsup_{n\to\infty} a_n = \inf_{n\geq 1} T_n = \infty.$$

Note that as long as we allow $\pm\infty$ as acceptable values of liminf and limsup they always exist. This is not true for limits, even if we allow $\pm\infty$. (Think of $(-1)^n$.)

Finally, for notational brevity we sometimes write $\liminf = \underline{\lim}$ and $\limsup = \overline{\lim}$.