Note Title 6/10/2015

Monotone Functions

Proposition 1.19

Let $f:(a,b)\to\mathbb{R}$ be monotone and let $a<c<b$. Then $f(c^-)$ and $f(c^+)$ both exist. Thus f can have only jump discontinuities.

Proof

We take the case where f is increasing. (The case where f is decreasing can be handled similarly, or we can notice that if f is decreasing then $-f$ is increasing.) In this case, $f(c)$ is an upper bound for $\{f(t) : a < t < c\}$, and a lower bound for

$$\{f(t) : c < t < b\}.$$

According to our Least Upper Bound Axiom this means $\sup\limits_{a<t<c} f(t)$ and $\inf\limits_{c<t<b} f(t)$ both exist as finite values. We claim that the first is $\lim\limits_{x \to c^-} f(x)$ and the second is $\lim\limits_{x \to c^+} f(x)$.

For the first, let $\varepsilon > 0$, and notice that there must exist some x_0 with $a < x_0 < c$ so that

$$\sup_{a<t<c} f(t) - \varepsilon < f(x_0) \leq \sup_{a<t<c} f(t).$$

Set $\delta = c - x_0 > 0$. Then if

$$c - \delta < x < c \quad (\text{i.e., } c - x < \delta)$$

we get $x_0 < x < c$ and so

$$f(x_0) \le f(x) \le \sup_{a<t<c} f(t).$$

Thus $\displaystyle \sup_{a<t<c} f(t) - f(x)$

$$\le \sup_{a<t<c} f(t) - f(x_0) < \varepsilon.$$

We can express this as

$$c-\delta < x < c \implies \sup_{a<t<c} f(t) - f(x) < \varepsilon.$$

This is the ε-δ definition of

$$\lim_{x \to c^-} f(x) = \sup_{a<t<c} f(t).$$

The analysis of $\inf_{c<t<b} f(t)$ is similar.