Note Title 6/11/2015

# Infinite Subsets of $\mathbb{N}$ are Countable

## Lemma 2.2

An infinite subset of $\mathbb{N}$ is countable; that is, if $A \subset \mathbb{N}$ and if $A$ is infinite, then $A$ is equivalent to $\mathbb{N}$.

## Proof

Recall that $\mathbb{N}$ is well-ordered in the following sense: each non-empty subset of $\mathbb{N}$ has a smallest element (since $\mathbb{N}$ is bounded below by 1). Here, $A$ has an infinite number of elements, so it's certainly non-empty. Denote its smallest element $x_1$, and consider the set $A \setminus \{x_1\}$. Denote the smallest element of this set $x_2$. Continuing in this way, we find an ordered sequence

$$x_1, x_2, x_3, \ldots$$

Notice in particular that the set $\{x_1, x_2, \dots\}$ will be infinite (because $A$ is).

The question is: are we assured of getting every element of $A$ by this process?

Suppose $A \setminus \{x_1, x_2, \dots\} \neq \emptyset$, and let $x \in A \setminus \{x_1, x_2, \dots\}$. We claim that there exists some $k$ so that $x_k > x$.

To see this, notice that by definition we cannot have $x = x_k$, and if $x_k < x$ $\forall k$ then $\{x_1, x_2, \dots\}$ would only have a finite number of values (and we've seen that it's infinite).

This means the set $\{k : x_k > x\}$ is non-empty and so has a least element, say $n$.

We have:

$$x_1 < x_2 < \ldots < x_{n-1} < x < x_n .$$

But this contradicts our definition of $x_n$ as the smallest element of

$$A \setminus \{x_1, x_2, \ldots, x_{n-1}\}. \qquad \square$$