Note Title 6/11/2015

Sequences in $\mathbb{R}$ have Monotone Subsequences

Theorem 2.3

Every sequence of real numbers has a monotone subsequence.

Proof Given any sequence of real numbers (a_n) we define $S = \{n : a_n < a_m \ \forall m > n\}$.

For example, suppose our starting sequence is

$$1, 3, 2, 4, 5, 7, 6, 8, \ldots$$
$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8, \ldots$$

$$S = \{1, 3, 4, 5, 7, 8, \ldots\}$$

As a second example, let's start with

$$7, 8, 9, 9, 10, 11, 11, \ldots,$$
$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

with the 11's repeating.

$$S = \{1, 2, 4, 5\}$$

If S is infinite, with elements

$$n_1 < n_2 < n_3 < \dots$$

then we must have

$$a_{n_1} < a_{n_2} < a_{n_3} < \dots,$$

and this is a strictly increasing subsequence (as with our first example).

If S is finite, then $\mathbb{N} \setminus S$ will be infinite, and we can find elements in $\mathbb{N} \setminus S$

larger than any elements in S. Let n_1 denote the least such element of $\mathbb{N} \setminus S$ (this would be 6 in our second example). Since $n_1 \notin S$ there is some n_2 so that $n_2 > n_1$ and $a_{n_2} \leq a_{n_1}$ (by definition of S. (E.g. $n_2 = 7$ would work in our second example.) But $n_2 \notin S$ and so there is some $n_3 > n_2$ so that

$a_{n_3} \leq a_{n_2}$. Continuing in this way, we create a decreasing subsequence

$$a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \ldots \qquad \square$$