F. Bernstein's Theorem

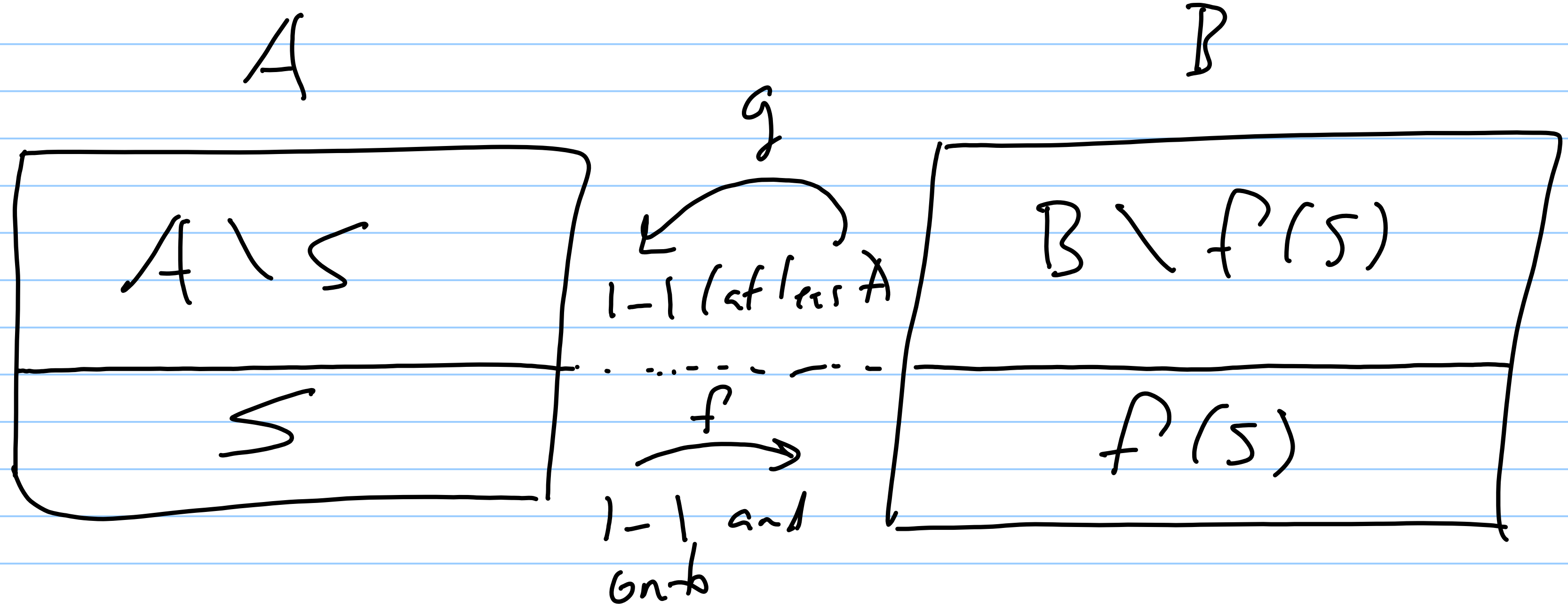
Note Title 6/12/2015

Theorem 2.13 (F. Bernstein's Theorem)

Let A and B be non-empty sets. If there exists a 1-1 map $f: A \to B$ and a 1-1 map $g: B \to A$ then there is a map $h: A \to B$ that is 1-1 and onto.

Proof

Carothers clarifies the basic idea with a figure. First, the idea is to find some $S \subset A$ so that we can define h to be f on S and g^{-1} on $A \setminus S$.



We need g (from $B \setminus f(S)$) to be onto $A \setminus S$. I.e., we need $g(B \setminus f(S)) = A \setminus S$.

To see this, we define a map

$$H : P(A) \to P(A) \text{ by}$$

$$\to H(S) = A \setminus g(B \setminus f(S))$$

for all $S \subset A$. Our goal is to find a fixed point of H, which is a set S so that $H(S) = S$. This will give $S = A \setminus g(B \setminus f(S))$.

Returning to our picture for A we'll have:

$$\begin{array}{c} A \\ \boxed{\begin{array}{c} g(B \setminus f(S)) = A \setminus S \\ \hline S = A \setminus g(B \setminus f(S)) \end{array}} \end{array}$$

<u>Claim</u> H is "increasing" in the following sense: $S \subset T \Rightarrow H(S) \subset H(T)$.

To see this, notice that

$$B \setminus f(T) \subset B \setminus f(S)$$

because we take away more on the left-hand side. So

$$g(B \setminus f(T)) \subset g(B \setminus f(S))$$

$$\Rightarrow A \setminus g(B \setminus f(S)) \subset A \setminus g(B \setminus f(T))$$

because we take away more with $g(B \setminus f(S))$.
Recalling the definition of $H(S)$ we get the claim.

Let
$$\mathcal{C} = \{ S \subset A : S \subset H(S) \}.$$
Now $\mathcal{C}$ is a collection of sets, so it makes sense to take its union, and we denote this $\bar{S} = \bigcup \mathcal{C}$.

We claim $H(\bar{S}) = \bar{S}$. We'll show this by establishing both $H(\bar{S}) \subset \bar{S}$ and $\bar{S} \subset H(\bar{S})$.

Beginning with the latter, notice that

$$S \subset \bar{S} \quad \forall\, S \in C \quad \text{(by definition)}$$

and $\forall\, S \in C \quad S \subset H(S) \subset H(\bar{S})$.

But if $S \subset H(\bar{S}) \ \forall\, S \in C$, if we take the union of these sets we are still ~~w~~ contained in $H(\bar{S})$. $\Rightarrow \bar{S} \subset H(\bar{S})$.

On the other hand, according to our assertion that H is increasing, we have

$$H(\bar{S}) \subset H(H(\bar{S}))$$

and this implies $H(\bar{S}) \in \mathcal{C}$. But $\bar{S}$ is the union of all sets in $\mathcal{C}$, so this means $H(\bar{S}) \subset \bar{S}$. $\square$