The Extended Cantor Function is Continuous

Corollary 2.19

The extended Cantor function $f:[0,1]\to[0,1]$ is continuous.

Proof

We've already established that the extended Cantor function is monotonic and onto, and so it follows from Corollary 2.18 that it's continuous. $\square$

Given any countable set D in $\mathbb{R}$ we can construct an increasing function $f:\mathbb{R}\to\mathbb{R}$ that is discontinuous precisely at the points of D. To see this, let

$$D = \{x_1, x_2, x_3, \ldots\}.$$

Let (ε_n) be a sequence of positive numbers so that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$.

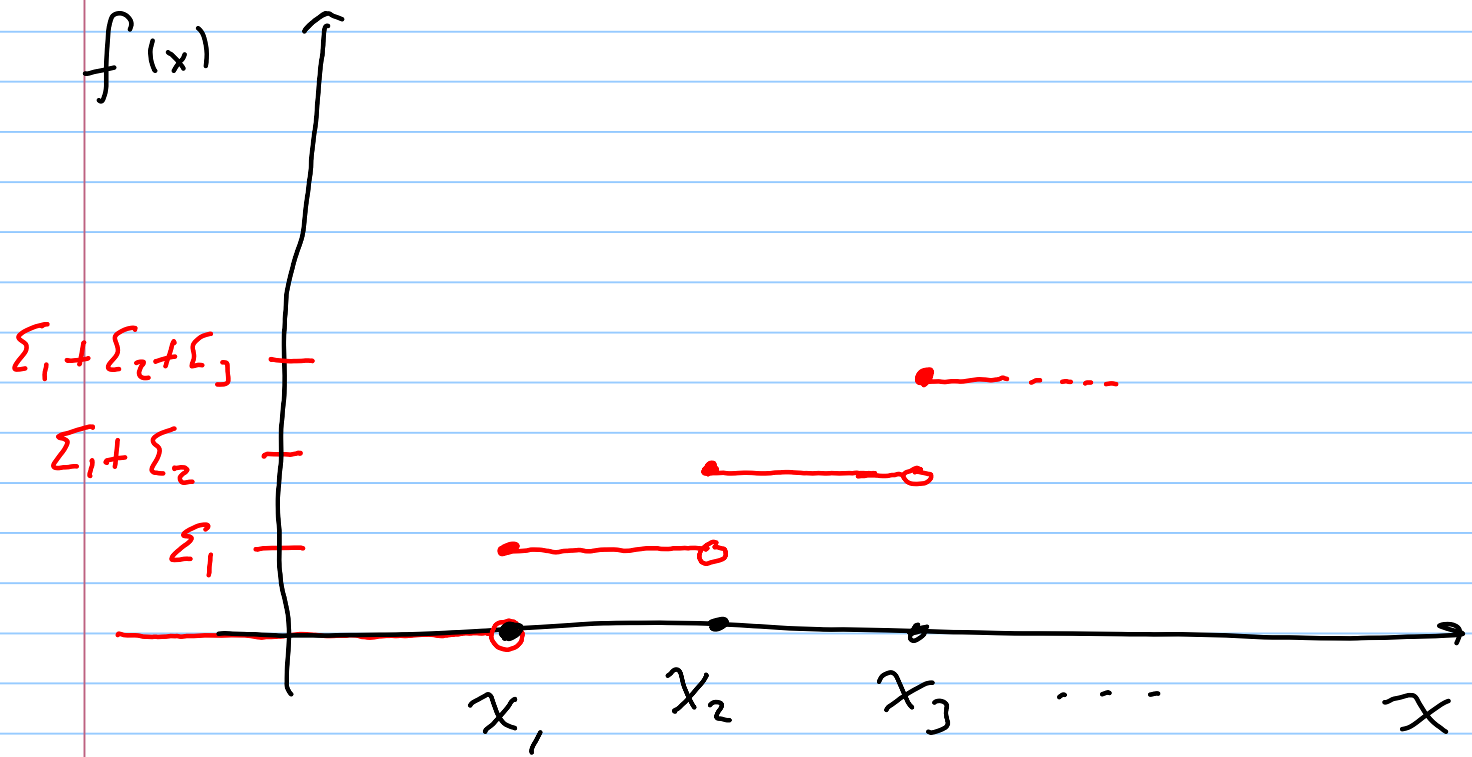
Set

$$f(x) := \sum_{x_n \le x} \varepsilon_n,$$

where the notation means to sum over $\{n : x_n \le x\}$.

Keep in mind: $f(x) := \sum_{x_n \leq x} \varepsilon_n$

A graph of f would look something like:



Notice that

$$0 \le f(x) \le \sum_{n=1}^{\infty} \varepsilon_n < \infty.$$

If $x < y$,

$$f(y) = \sum_{x_n \le y} \varepsilon_n = \sum_{x_n \le x} \varepsilon_n + \sum_{x < x_n \le y} \varepsilon_n$$

$$= f(x) + \sum_{x < x_n \le y} \varepsilon_n \ge f(x)$$

$\Rightarrow f$ is non-decreasing (increasing).

Two things are left to show:

1. f is discontinuous at the points of D.

2. f is continuous at the points of $\mathbb{R} \setminus D$.

We'll see that these are true in Problem 2.34.