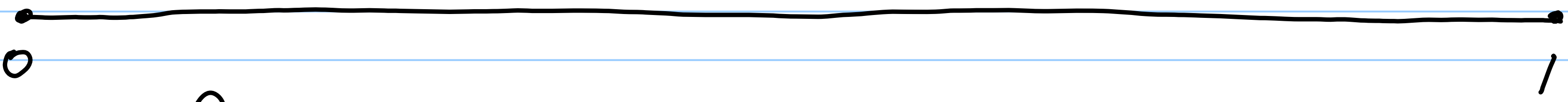
The Cantor Set

Note Title 6/13/2015

Here's how we construct the Cantor set:

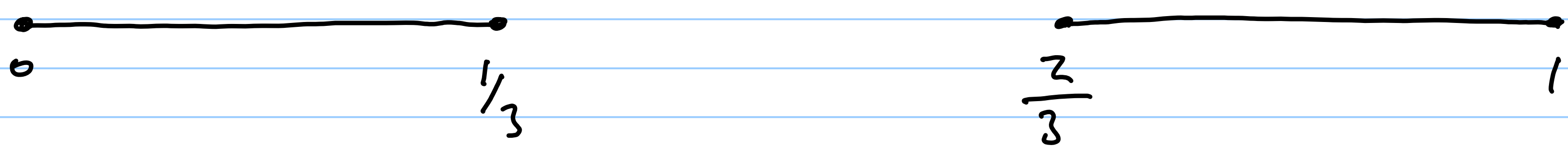
We start with the interval $[0, 1]$:

I_0 :



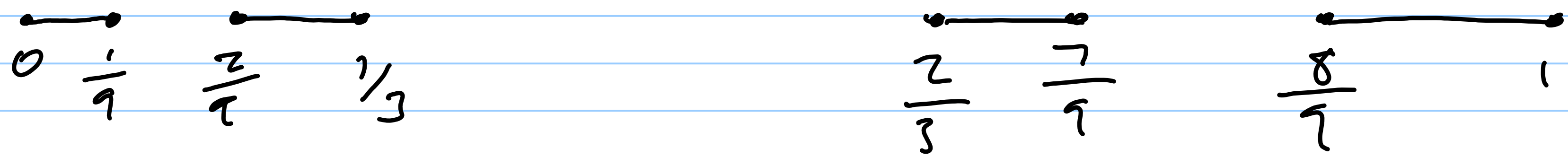
At the first step of the process we remove the open middle third interval $\left(\frac{1}{3}, \frac{2}{3}\right)$

I_1 :



In the next step we remove the open middle third of each of these remaining intervals:

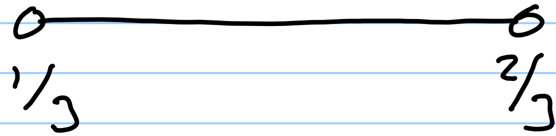
I_2 :



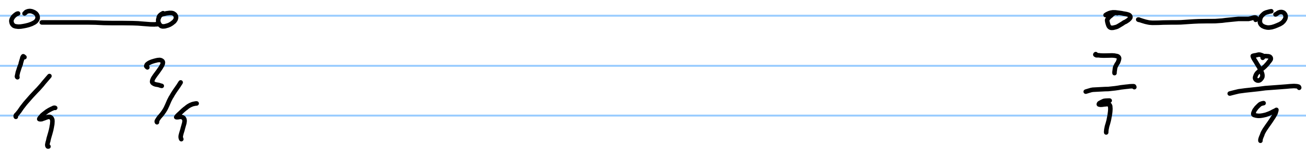
Notice that at each step of the process we construct I_n from I_{n-1} $(n = 1, 2, \ldots)$ by removing 2^{n-1} disjoint open middle thirds from I_{n-1}, each of length 3^{-n}.

Carothers denotes the discarded open sets J_n, $n = 1, 2, \ldots$ That is:

J_1 :



J_2 :



etc.

The Cantor set is the set that remains at the end of this process (what's left after all the J_n's are taken away).

We can express this as

$$\Delta = \bigcap_{n=1}^{\infty} I_n$$

Cantor set

Since Δ contains all the interval endpoints we see that it is at least countably infinite:

$$0, \; 1, \; \underbrace{\frac{1}{3}, \; \frac{2}{3}}_{\text{step } 1}, \; \underbrace{\frac{1}{9}, \; \frac{2}{9}, \; \frac{7}{9}, \; \frac{8}{9}}_{\text{step } 2}, \; \ldots$$

I.e., we get 2^n additional points at each step. We refer to these values as the endpoints of Δ. Notice that they must have the form $\frac{a}{3^n}$ for $a, n \in \mathbb{N}$ $a \leq 3^n$.