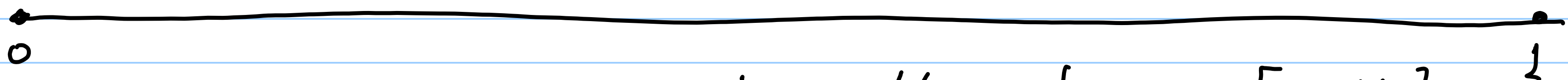
Note Title 6/13/2015

Cardinality of the Cantor Set

Claim: The Cantor is uncountable, and in fact $\operatorname{card}(\Delta) = c$.

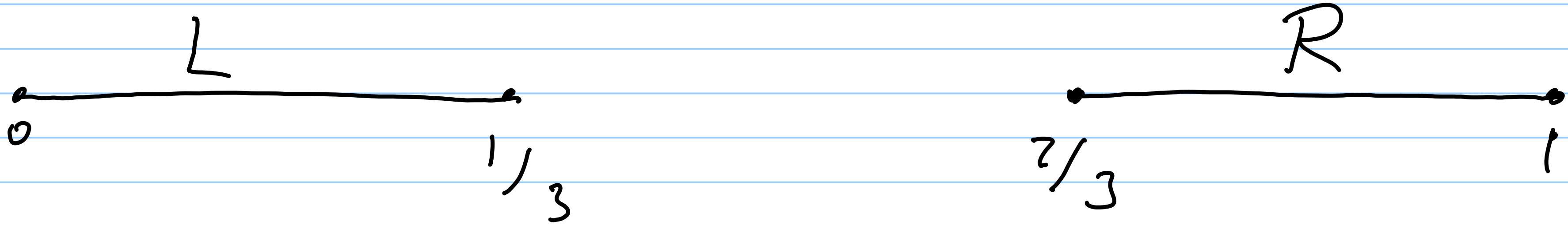
To see this, let's think again about how we construct the Cantor set. We start with $[0,1]$:

I_0

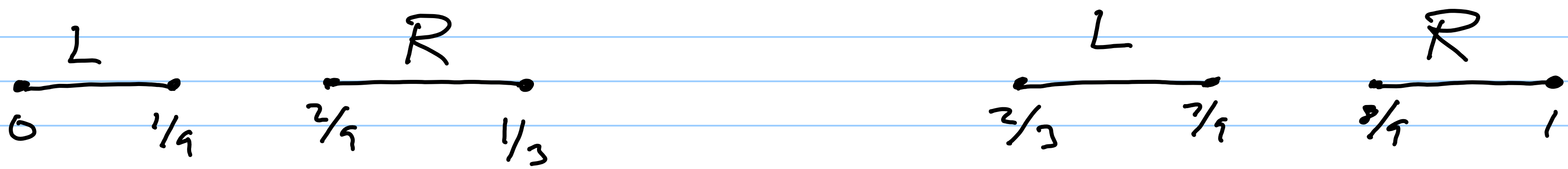


Every point in Δ must either be in $[0, \frac{1}{3}]$ or $[\frac{2}{3}, 1]$.

We can designate these by two choices, left L or right R:

I_1 : 

Likewise, every point in $\Delta \cap [0, 1/3]$ is either in $[0, 1/9]$ or $[2/9, 1/3]$, which we again will designate as left L or right R:

I_2 : 

At this point, the sequence LL corresponds with all elements of Δ in $[0, 1/9]$, the sequence LR corresponds with all elements of Δ in $[2/9, 1/3]$ etc.

If we continue this process, we see that every element of the Cantor set corresponds with an infinite sequence of L's and R's.

But of course the number of sequences of L's and R's is in 1-1 correspondance with the number of sequences of 0's and 1's, and we already know the sequences of 0's and 1's has cardinality c. We see that

$$\mathrm{Card}(\Delta) = c.$$

On the other hand, Δ is small in the following sense: Let's measure the sizes of the sets taken away:

$$|J_1| = 1/3, \quad |J_2| = \frac{2}{9}, \quad |J_3| = \frac{4}{3^3}$$

$$\ldots \quad |J_n| = \frac{2^{n-1}}{3^n}, \quad n = 1, 2, \ldots$$

The sizes of all these intervals is

$$\frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \frac{1}{1 - 2/3} = 1,$$

using a geometric series. It appears that the intervals we've taken away have the same size as the interval we started with. So

we have:

$$\operatorname{card}(\Delta) = c \quad \text{but} \quad |\Delta| = 0.$$