The Cantor Function

Note Title 6/14/2015

As a starting point, recall from Theorem 2.15 that if $x \in \Delta$ we can express it as

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

with a_n taking values 0 and 2. If we write $b_n = \frac{a_n}{2}$, we can express this as

$$x = \sum_{n=1}^{\infty} \frac{2b_n}{3^n}, \quad \text{where } b_n = 0 \text{ or } 1.$$

We define the Cantor function as the map

$f : \Delta \to [0,1]$ by

$$f\left(\sum_{n=1}^{\infty} \frac{2b_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$$

That is,

$$f(.a_1 a_2 a_3 \ldots \text{ (base 3)}) = .\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\ldots \quad \text{(base 2)}$$

For example,

$$f(.2 \text{ base } 3) = .\frac{2}{2} = .1 \quad \text{(base 2)}$$

Since the domain of f includes all possible sequences for which the a_n take values 0 or 2, we get (in the range of f) all binary expansions, which gives all of $[0,1]$ by Proposition 1.8. We can conclude that f is onto.

We claim, however, that f is not 1-1.

For example,

$$f(1/3) = f(.0\overline{2} \text{ base } 3)$$

$$= .0\overline{1} \text{ base } 2 = 1/2$$

On the other hand

$$f(2/3) = f(.2 \text{ base } 3) = .1 \text{ base } 2 = 1/2$$

We conclude that f is not 1-1.

Finally, let's check that f is non-decreasing.

I.e., $x < y \Rightarrow f(x) \leq f(y)$.

This is easy to see because the values of x and y and $f(x)$ and $f(y)$ are determined by the same values (a_n).

I.e., if $x = .a_1 a_2 a_3 \ldots$ (base 3)

and $y = .\tilde{a}_1 \tilde{a}_2 \tilde{a}_3 \ldots$ (base 3)

Then $x < y$ means $a_1 < \hat{a}_1$ or $a_1 = \hat{a}_1$ and

$a_2 < \tilde{a}_2$ or $a_1 = \tilde{a}_1$, $a_2 = \tilde{a}_2$ and $a_3 < \hat{a}_3$ etc. In all of these cases,

$$f(x) = .\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\ldots \quad (\text{base } 2)$$

will be less than or equal to

$$f(y) = .\frac{\tilde{a}_1}{2}\frac{\hat{a}_2}{2}\frac{\hat{a}_3}{2}\ldots \quad (\text{base } 2)$$