Extension of the Cantor Function

Note Title 6/14/2015

At this point, the Cantor function is only defined on Δ, and not on any of the deleted intervals J_1, J_2, etc. We'll fix this by defining the Cantor function on $[0,1]$ as a continuous function that is constant between the endpoints of Δ.

As a start, recall that
$$f(1/3) = f(2/3) = 1/2.$$

We define $f(x) = 1/2$ for $\frac{1}{3} < x < \frac{2}{3}$.

Likewise, $f(1/9) = f(2/9) = 1/4$, so we set $f(x) = 1/4$ for $\frac{1}{9} < x < 2/9$.

We can express this generally as
$$f(x) = \sup\{f(y) : y \in \Delta,\ y \leq x\}.$$

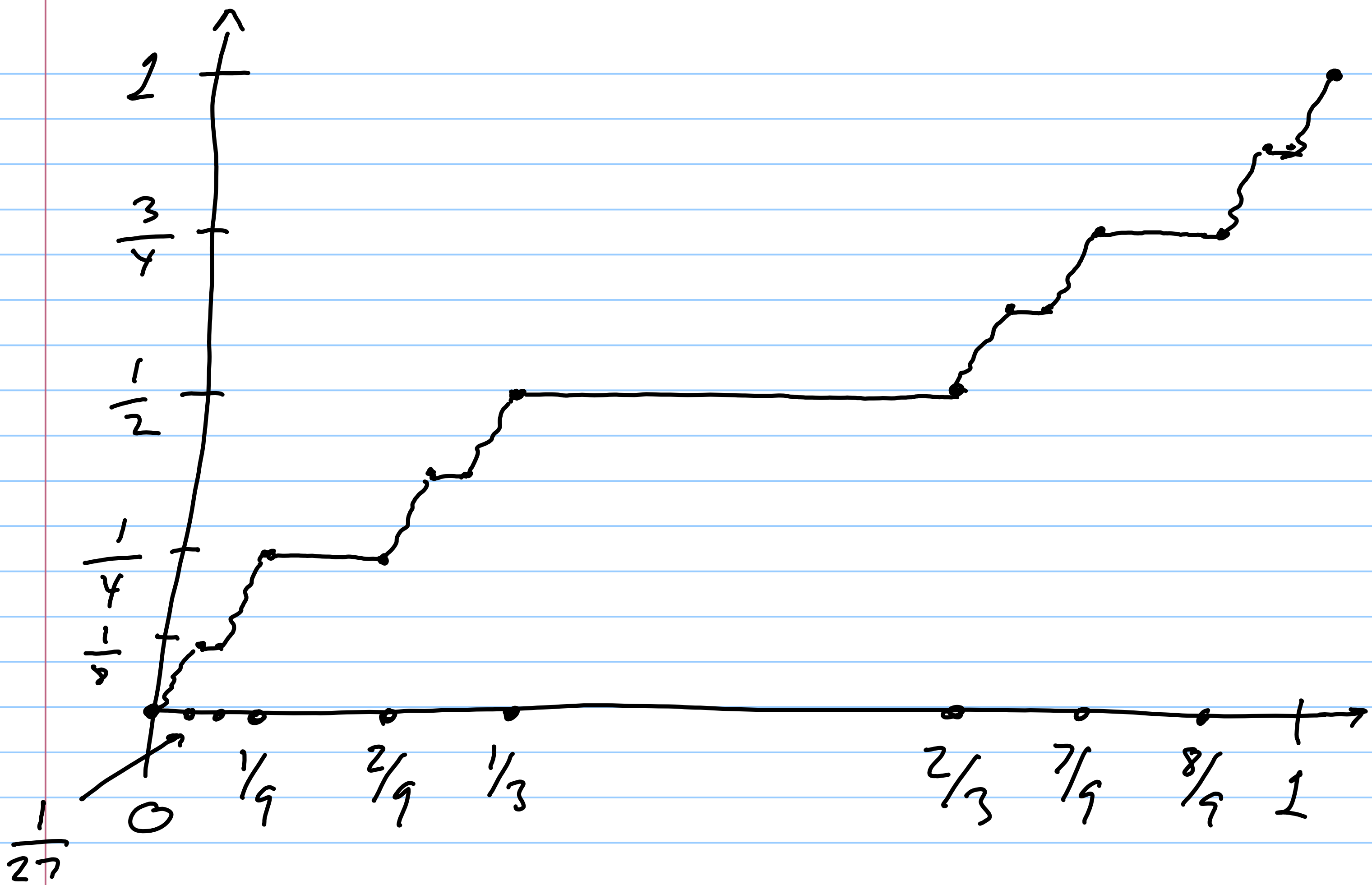
For example, suppose we want to evaluate $f(1/2)$.

We compute

$$f(1/2) = \sup\{f(y) : y \in \Delta,\ y \le 1/2\}$$

$$= f(1/3) = 1/2.$$

The extended Cantor function is a bit tricky to draw, but let's try on the next page.


1
3/4
1/2
1/4
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1/27
0
1/9
2/9
1/3
2/3
7/9
8/9
1

Continuing in this way, we fill in a continuous (continuity will be proved in the next section). This extended Cantor function is sometimes called the Cantor–Lebesgue function or Lebesgue's singular function, though Carothers refers to it as the extended Cantor function, and sometimes as the Cantor function.