Some Properties of Monotonic Functions

Note Title 6/15/2015

In many cases it's productive to work with functions that aren't quite continuous, and it turns out that monotonic functions are often "good enough." One of the primary reasons for this is the following theorem.

Theorem 2.17

If $f : (a,b) \longrightarrow \mathbb{R}$ is monotonic, then f has at most countably many points of discontinuity in (a,b), all of which are jump discontinuities.

Proof

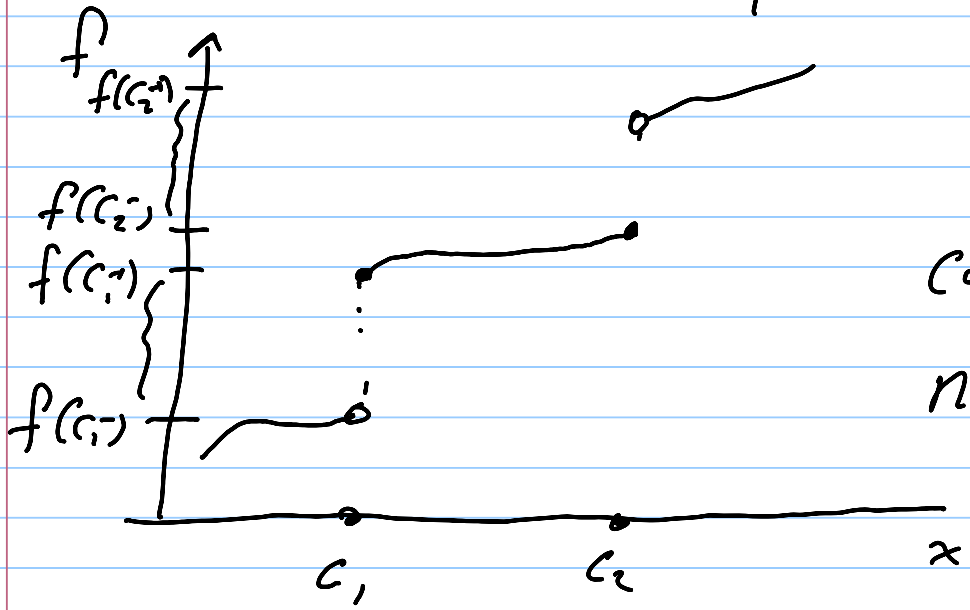
First, the remark that all discontinuities are jump discontinuities follows from Proposition 1.19.

Let's assume f is non-decreasing (a similar argument holds if f is non-increasing).

If f has a discontinuity at c, we must have $f(c^-) < f(c^+)$ (because of monotonicity, and if $f(c^-) = f(c^+)$ then f is continuous at c). This defines an interval $(f(c^-), f(c^+))$.

Likewise, we can define the collection of intervals

$$\mathcal{I} := \{ (f(c^-), f(c^+)) : \text{where } c \text{ is a point of discontinuity of } f \}$$



We see that $\mathcal{I}$ is a collection of pairwise disjoint nonempty intervals, and so must be countable by Problem 2.15. $\square$

Corollary 2.18

If $f : [a,b] \to [c,d]$ is both monotone and onto then it's continuous.

Proof

This is because a jump discontinuity would lead to an interval $(f(c^-), f(c^+))$ not obtained in $[c,d]$. $\square$