$\mathbb{R}^n$ with the Euclidean Metric, and ℓ_2

Note Title 7/4/2015

For $\mathbb{R}^n$ with

$$d(x,y) = \|x-y\|_2 = \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{1/2}$$

we have a bit of notational difficulty, because we need to denote sequences for which each element has components. Following Carothers, we use the notation $(x^{(k)})$ as the sequence where

$$x^{(k)} = (x_1^k, x_2^k, \ldots, x_n^k)$$

In Problem 3.18 we'll see that

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le n\|x\|_\infty.$$

From these inequalities, we see that a sequence $(x^{(k)})$ converges in $\mathbb{R}^n$ iff each of its components converges in $\mathbb{R}$.

This allows us to conclude immediately that Cauchy sequences in $\mathbb{R}^n$ converge in $\mathbb{R}^n$, and that bounded sequences in $\mathbb{R}^n$ have subsequences that converge in $\mathbb{R}^n$.

For ℓ_2, we have $\|x\|_\infty \le \|x\|_2$, which means that convergence in ℓ_2 implies coordinate-wise convergence. But the converse is false.

For example, consider the sequence $(e^{(k)})$ defined by

$$e^{(k)} = (0, 0, \dots, 0, \underset{\substack{\uparrow \\ k^{\text{th}} \text{ slot}}}{1}, 0, \dots, 0)$$

for $k = 1, 2, \dots$ Clearly

$$e_j^k \longrightarrow 0 \qquad \text{as } k \to \infty$$

for each j. E.g. $e_1^1 = 1$, $e_1^2 = 0$, $e_1^3 = 0$, ...

But $\|e^{(k)}\|_2 = 1$ for all k. So certainly

$\|e^{(k)}\|_2$ is not converging to 0.