The Cauchy-Schwarz Inequality For ℓ_2

Note Title 7/1/2015

Lemma 3.3 (Cauchy-Schwarz inequality for ℓ_2)

For any $x, y \in \ell_2$ we have

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_2 \|y\|_2 .$$

Proof

In anticipation of our discussion of inner products, we'll set $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$

Notice that $\langle x, x \rangle = \|x\|_2^2$.

As a start, suppose that x and y each have only a finite number of non-zero terms, and that neither is 0. (The case in which either is identically 0 is trivial.) In this case, we have

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

Given any $t \in \mathbb{R}$, notice that

$$0 \le \|x+ty\|_2^2 = \langle x+ty, x+ty \rangle$$

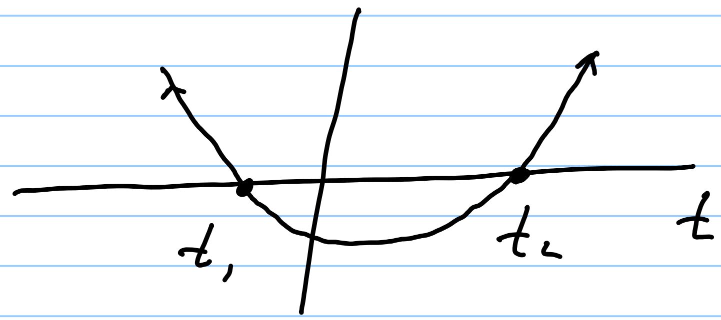
$$= \langle x, x \rangle + 2t\langle x, y \rangle + t^2 \langle y, y \rangle$$

$$= \|x\|_2^2 + 2t\langle x, y \rangle + t^2 \|y\|_2^2 .$$

In the case of equality, we can solve with the quadratic formula to get

$$t = \frac{-2\langle x,y\rangle \pm \sqrt{4\langle x,y\rangle^2 - 4\|x\|_2^2\|y\|_2^2}}{2\|y\|_2^2}.$$

If there were two real roots for this polynomial (which opens upward) we could have values of t for which it is negative:



This means we must have

$$4\langle x,y\rangle^2 - 4\|x\|_2^2\|y\|_2^2 \le 0.$$

This implies $\langle x,y\rangle^2 \le \|x\|_2^2\|y\|_2^2$, which gives $|\langle x,y\rangle| \le \|x\|_2\|y\|_2$. If we now apply this to the sequences $(|x_i|)$, $(|y_i|)$ we get

$$\sum_{i=1}^{n} |x_i||y_i| \le \|x\|_2\|y\|_2.$$

Now if (x_i) and (y_i) are general sequences (i.e., with an infinite number of non-zero terms) we still have this last inequality because the right-hand side is simply bigger. The inequality is true $\forall n$, so we can take $n \to \infty$ to get the claim. $\square$