Norms on l_p Spaces, II

Note Title 7/3/2015

Lemma 3.7 (Hölder's inequality for l_p spaces)

Let $1 < p < \infty$ and let q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. Given $x \in l_p$ and $y \in l_q$ we have

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \|x\|_p \|y\|_q .$$

Proof First, we can take $\|x\|_p, \|y\|_q > 0$, because otherwise the inequality is trivial.

Now we can use Young's inequality to write:

$$\sum_{i=1}^{n} \left| \frac{x_i y_i}{\|x\|_p \|y\|_q} \right| \le \sum_{i=1}^{n} \left(\frac{1}{p} \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \frac{|y_i|^q}{\|y\|_q^q} \right)$$

$$= \frac{1}{p\|x\|_p^p} \underbrace{\sum_{i=1}^{n} |x_i|^p}_{\le \|x\|_p^p} + \frac{1}{q\|y\|_q^q} \underbrace{\sum_{i=1}^{n} |y_i|^q}_{\le \|y\|_q^q}$$

$$\le \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \frac{1}{\|x\|_p \|y\|_q} \sum_{i=1}^{n} |x_i y_i| \le 1$$

$$\Rightarrow \sum_{i=1}^{n} |x_i y_i| \le \|x\|_p \|y\|_q$$

Taking $n \to \infty$ we get the claim. $\square$

Theorem 3.8 (Minkowski's Inequality for ℓ_p)

Let $1 \le p \le \infty$. If $x, y \in \ell_p$ then $x + y \in \ell_p$ and

$$\|x+y\|_p \le \|x\|_p + \|y\|_p.$$

Proof

First, we notice that this is easy for $p=1$ and $p=\infty$. For $p=1$, we have

$$\sum_{i=1}^{n} |x_i + y_i| \le \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$

$$\le \|x\|_1 + \|y\|_1$$

Taking $n \to \infty$ we get

$$\|x+y\|_1 \le \|x\|_1 + \|y\|_1 .$$

For $p = \infty$,

$$\|x+y\|_\infty = \sup_n |x_n + y_n| \le \sup_n |x_n| + \sup_n |y_n|$$

$$= \|x\|_\infty + \|y\|_\infty$$

For $1 < p < \infty$, we start by noticing that if $x \in \ell_p$ then $(|x_n|^{p-1}) \in \ell_q$. This is because

$$\|(|x_n|^{p-1})\|_q^q = \sum_{n=1}^{\infty} (|x_n|^{p-1})^q = \sum_{n=1}^{\infty} |x_n|^{q(p-1)}$$

$$= \underbrace{\sum_{n=1}^{\infty} |x_n|^p}_{\|x\|_p^p} < \infty$$

$$\Longrightarrow \boxed{\|(|x_n|^{p-1})\|_q = \|x\|_p^{p/q}}$$

Now write:

$$\|x+y\|_p^p = \sum_{i=1}^{\infty} |x_i+y_i|^p = \sum_{i=1}^{\infty} |x_i+y_i|\,|x_i+y_i|^{p-1}$$

$$\le \sum_{i=1}^{\infty} |x_i|\,|x_i+y_i|^{p-1} + \sum_{i=1}^{\infty} |y_i|\,|x_i+y_i|^{p-1}$$

$$\le \|x\|_p \,\|(|x_n+y_n|^{p-1})\|_q + \|y\|_p \,\|(|x_n+y_n|^{p-1})\|_q$$

Hölders inequality

box

$$= \|x\|_p \,\|x+y\|_p^{p/q} + \|y\|_p \,\|x+y\|_p^{p/q}$$

$$= \|x+y\|_p^{p/q} \left(\|x\|_p + \|y\|_p \right)$$

$\Rightarrow$

$$\|x+y\|_p^{p} \leq \|x+y\|_p^{p/q} \left(\|x\|_p + \|y\|_p \right)$$

$$\Rightarrow \|x+y\|_p^{p - \frac{p}{q}} \leq \|x\|_p + \|y\|_p$$

But $p - \frac{p}{q} = 1$ by our definition of q as a Hölder conjugate of p. That is,

$$\frac{1}{p} + \frac{1}{q} = 1. \Rightarrow 1 + \frac{p}{q} = p \Rightarrow p - \frac{p}{q} = 1. \qquad \square$$