Limits in Metric Spaces

Note Title 7/4/2015

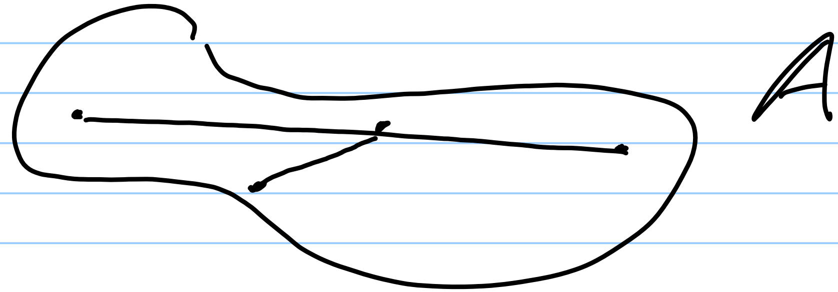
Given $x \in M$ and $r > 0$ we can express an open ball centered at x with radius r as

$$B_r(x) = \{y \in M : d(x,y) < r\}.$$

We say $A \subset M$ is bounded if it is contained in some ball; i.e., if $A \subset B_r(x)$ for some $x \in M$, $r > 0$.

We define the diameter of a set A to be

$$\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\}.$$



By a <u>neighborhood</u> of x Carothers means any set containing an open ball centered at x. Most authors take neighborhoods to be open. (Open and closed sets will be discussed in Chapter 4.)

Definition (Convergence in a metric space)

We say that a sequence of points $(x_n) \subset M$ converges to a point $x \in M$ if

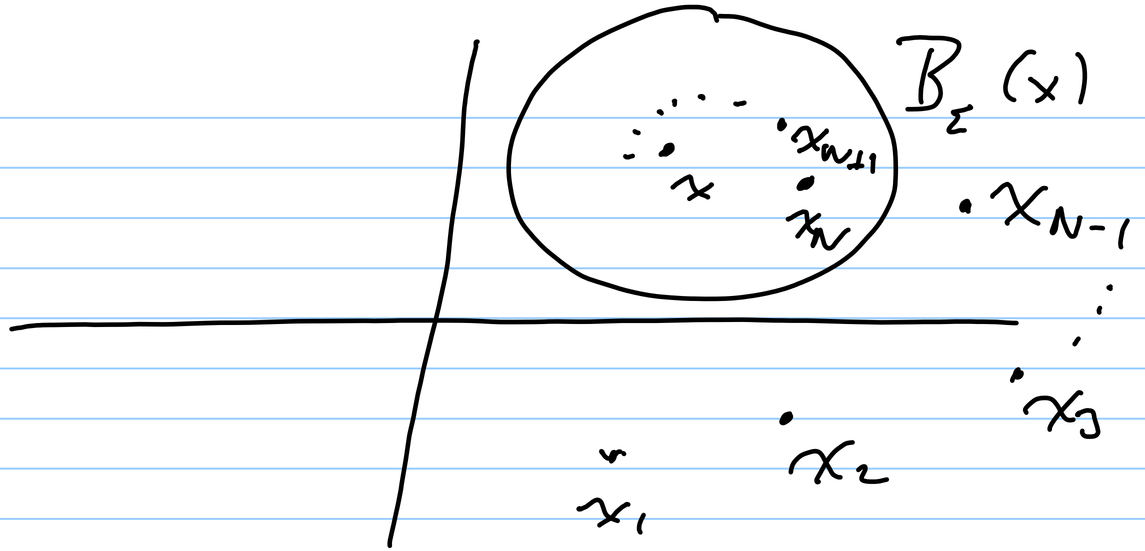
$$\lim_{n \to \infty} d(x_n, x) = 0.$$

If we express this in terms of the definition of a limit, it says that (x_n) converges to x if given any $\varepsilon > 0$ there exists an integer N so that

$$n \geq N \implies d(x_n, x) < \varepsilon.$$

Equivalently, we could say that (x_n) converges to x if given any $\varepsilon > 0$ there exists an integer N so that

$$n \geq N \implies \{x_n : n \geq N\} \subset B_\varepsilon(x).$$



Slightly more generally, if

$$\{x_n : n \geq N\} \subset A$$

for some N, we say the sequence (x_n) is __eventually in A__.

With this terminology, we can say that (x_n) converges to x if and only if (x_n) is eventually in every neighborhood of x.