Cauchy Sequences for Metric Spaces

Note Title 7/4/2015

<u>Definition</u> (Cauchy sequences for metric spaces)

We say a sequence $(x_n) \subset M$ is Cauchy if given any $\varepsilon > 0$ there exists N so that

$$m, n \geq N \implies d(x_n, x_m) < \varepsilon.$$

I.e.,

$$\operatorname{diam}(\{x_n : n \geq N\}) \leq \varepsilon$$

Two properties of $\mathbb{R}$ with the absolute value metric are: (1) every Cauchy sequence in $\mathbb{R}$ converges to a point in $\mathbb{R}$; (2) every bounded sequence in $\mathbb{R}$ has a convergent subsequence (Bolzano - Weierstrass Theorem). It turns out that neither of these holds for all metric spaces. We'll say more about this in Chapter 7 on Complete Metric Spaces.

Example

Consider $M = (0,1]$, with the absolute value metric, and the sequence $(x_n) = \left(\frac{1}{n}\right)$. The sequence converges to 0 as $n \to \infty$, but $0 \notin M$, so it doesn't converge to an element of M, even though it's Cauchy. Moreover, any subsequence converges to 0, so no subsequence converges to an element of M.