Unions and Intersections of Open Sets

Note Title 7/13/2015

Notice that by covering every point x in an open set U by some open ball $B_{\varepsilon(x)}(x) \subset U$ we can express U as

$$U = \bigcup_{x \in U} \{ B_{\varepsilon(x)}(x) : B_{\varepsilon(x)}(x) \subset U \}.$$

We can also turn this around as follows:

Theorem 4.3

An arbitrary union of open sets is again open; that is, if $(U_\alpha)_{\alpha \in A}$ is any collection of open sets then

$$V = \bigcup_{\alpha \in A} U_\alpha$$

is open.

Proof If $x \in V$ then $x \in U_\alpha$ for some U_α, and since U_α is open it contains a neighborhood containing x, which is also in V. $\square$

It's easy to see that the same is not true of infinite intersections. Consider the open sets $(0, 1+1/n)$, and notice that

$$\bigcap_{n=1}^{\infty} \left(0, 1+\frac{1}{n}\right) = (0, 1].$$

This set is not open in $\mathbb{R}$, because there is no $\varepsilon > 0$ small enough so that $B_\varepsilon(1) \subset (0,1]$. Here, $B_\varepsilon(1) = (1-\varepsilon, 1+\varepsilon)$. Nonetheless, we have:

Theorem 4.4

A finite intersection of open sets is open; that is, if each of $U_1, U_2, \ldots, U_n$ is open then so is $V = \bigcap_{i=1}^{n} U_i$.

Proof

If $x \in V$ then $x \in U_i \ \forall i$, and since each U_i is open, each has a ball $B(x, \varepsilon_i) \subset U_i$.

Let $\varepsilon = \min_i \varepsilon_i$, and notice that

$$B(x, \varepsilon) \subset B(x, \varepsilon_i)$$

$\forall i$. This implies $B(x, \varepsilon) \subset V$, and so serves as the necessary ball centered at x. $\square$