Note Title 7/13/2015

# Open Subsets of $\mathbb{R}$

## Theorem 4.6

If $U$ is an open subset of $\mathbb{R}$, then $U$ may be written as a countable union of disjoint open intervals. That is, $U = \bigcup_{n=1}^{\infty} I_n$, where $I_n = (a_n, b_n)$ (possibly unbounded) and $I_n \cap I_m = \emptyset$ for $n \neq m$.

## Proof

As discussed previously, by covering each point of $U$ with an interval, we can see that $U$ can be expressed as a union of intervals. The question is: Can we take these intervals to be disjoint? (If so, we get that they are countable from Problem 2.15.)

## Claim

For each $x \in U$ there is a maximal open interval $I_x \subset U$ so that if $I \subset U$ is any other open interval containing $x$ then $I \subset I_x$.

To see this claim, set

$$a_x := \inf \{ a : (a, x] \subset U \}$$

$$b_x := \sup \{ b : [x, b) \subset U \}.$$

These values exist, and so $I_x = (a_x, b_x)$ is well-defined. Also, if $x \in I$ and $I \subset U$ is open we must have $I \subset I_x$ or we could get a contradiction. (I.e., $I = (c, d)$, and we would have $c < a_x$ or $d > b_x$.)

Now for any $x, y \in U$ we either have

$$I_x \cap I_y = \emptyset \quad \text{or} \quad I_x = I_y.$$

This is simply because if $y \in I_x$ then the interval around $y$ can be expanded to exactly the same maximal interval as the interval around $x$.

We can now take $I_x$ at each $x \in U$ and throw out all duplicates to get

$$U = \bigcup_{x \in U} I_x \, ,$$

where by throwing out all duplicates we have a disjoint union. $\square$