Note Title 7/14/2015

Sequential Characterization of Open Sets

Theorem 4.7

A set U in (M, d) is open if and only if whenever a sequence $(x_n) \subset M$ converges to a point $x \in U$ we have $x_n \in U$ for all but finitely many n.

Proof $(\Longrightarrow)$ Suppose U is open. We know that if $x_n \to x$ then (x_n) is eventually in $B_\varepsilon(x)$ for every $\varepsilon > 0$, and since U is open this means (x_n) is eventually in U.

($\Leftarrow$) On the other hand, suppose that whenever $(x_n) \subset M$ converges to $x \in U$ we have $x_n \in U$ for all but finitely many n. We want to show that U must be open.

Suppose not (i.e., that U is not open). Then there is $x \in U$ so that $B_\varepsilon(x) \cap U^c \neq \emptyset$ for all $\varepsilon > 0$. (I.e., $B_\varepsilon(x)$ is not contained entirely in U.) In particular, for each n there is some $x_n \in B_{\frac{1}{n}}(x) \cap U^c$. But then $(x_n) \subset U^c$ and $x_n \to x$ ($\because |x_n - x| < \frac{1}{n}$). And this contradicts our assumption. $\square$