Sequential Characterization of Closed Sets

Note Title 7/14/2015

Recall that F is closed iff F^c is open, which means F is closed iff

$$x \in F^c \Longrightarrow B_\varepsilon(x) \subset F^c$$

for $\varepsilon > 0$ sufficiently small. This can be re-stated as follows: F is closed iff

$$B_\varepsilon(x) \cap F \neq \emptyset \;\; \forall \varepsilon > 0 \Longrightarrow x \in F.$$

For example, suppose we're checking if $F = (0,1)$ is closed. We see that

$$B_\varepsilon(0) \cap F \neq \emptyset$$

for any $\varepsilon > 0$, but $0 \notin F$. We conclude from our criterion that F is not closed.

Theorem 4.9

Given a set F in (M, d), the following are equivalent:

(i) F is closed; that is F^c is open.

(ii) If $B_\varepsilon(x) \cap F \neq \emptyset$ $\forall \varepsilon > 0$ then $x \in F$.

(iii) If a sequence $(x_n) \subset F$ converges to some point $x \in M$ then $x \in F$.

Note. It's condition (iii) that leads to the name "closed." We say the set is closed with respect to limits.

Proof

We did the case (i) $\Longleftrightarrow$ (ii) in the lead-in discussion. To see (ii) $\Longrightarrow$ (iii), let (ii) hold, and suppose $(x_n) \subset F$ is a sequence so that $x_n \to x$ in M. Then $B_\varepsilon(x)$ contains infinitely many x_n for any $\varepsilon > 0$, and so

$$B_\varepsilon(x) \cap F \neq \emptyset$$

for any $\varepsilon > 0$ (in fact, the intersection contains an infinite number of points). Thus $x \in F$ by (ii).

For (iii) $\Rightarrow$ (ii), let (iii) hold, and suppose that for some $x \in M$ $B_\varepsilon(x) \cap F \neq \emptyset$ $\forall \varepsilon > 0$. Then for each $n = 1, 2, \ldots$ there is $x_n \in B_{\frac{1}{n}}(x) \cap F$,

and so the sequence (x_n) converges to x.
But by (iii) this means $x \in F$.

We now have:

$$(i) \iff (ii) \iff (iii) \qquad \square$$