The Relative Metric

Note Title 7/15/2015

The object here is to consider the nature of open sets in a metric space (A,d) that is a subset of a metric space (M,d). As a start, we notice the relationship between a ball in A, which we denote $B_\varepsilon^A(x)$, and a ball in M, which we denote $B_\varepsilon^M(x)$. We have:

$$B_\varepsilon^A(x) = \{a \in A : d(x,a) < \varepsilon\}$$

$$= A \cap \{y \in M : d(x,y) < \varepsilon\}$$

$$= A \cap B_\varepsilon^M(x).$$

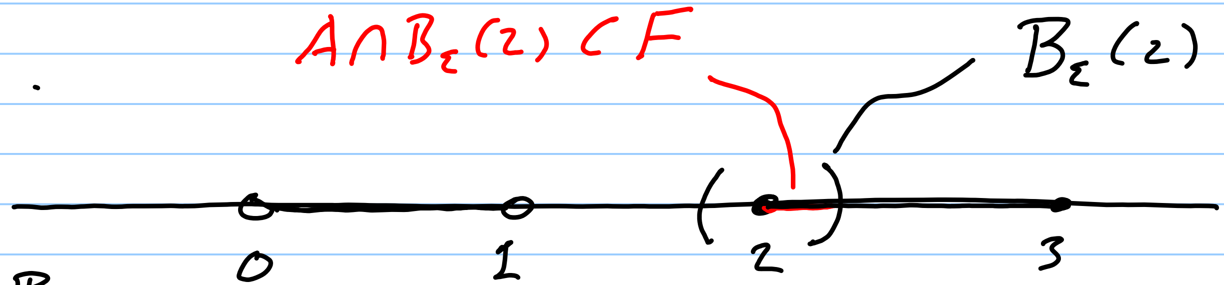
We see that a subset $G \subset A$ is open in (A,d) (i.e., open relative to A) if given any $x \in G$ there is some $\varepsilon > 0$ so that

$$A \cap B_\varepsilon^m(x) = B_\varepsilon^A(x) \subset G.$$

Examples

1. Suppose $M = \mathbb{R}$ and $A = (0,1) \cup [2,3]$.

The set $G = (0,1)$ is clearly open in (A,d), but $F = [2,3]$ is also open in A (i.e., relative to A). Let's check the endpoint $x = 2$.



$A \cap B^{\mathbb{R}}_\varepsilon(2) \subset F$ for $\varepsilon > 0$ sufficiently small.

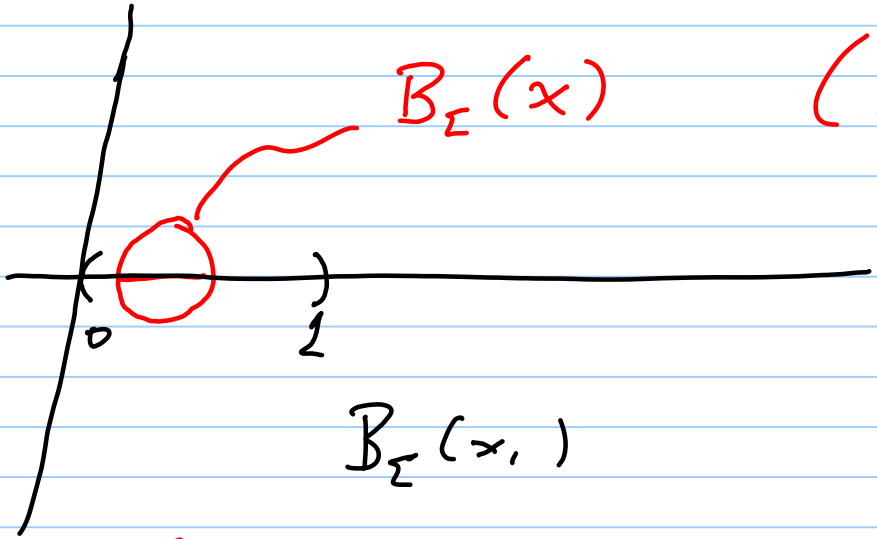
2. Take $M = \mathbb{R}^2$ and $A = \mathbb{R} \times \{0\}$.

The set $G = (0,1) \times \{0\}$ is not open in $\mathbb{R}^2$ (it contains no balls in $\mathbb{R}^2$), but is open relative to A. I.e., take any $x \in G$, which we can express as $x = (x_1, 0)$, with $x_1 \in (0,1)$. We need to find $\varepsilon > 0$ small enough so that

$$A \cap B_\varepsilon^{\mathbb{R}^2}(x) \subset G.$$

To do this take $\varepsilon > 0$ small enough so that

$$B_\varepsilon(x_1) \subset (0,1).$$



(recall: $x = (x_1, 0)$)

$$B_\varepsilon^{\mathbb{R}^2}(x) \cap A \subset G$$