Characterizations of Continuity

Note Title 7/17/2015

Theorem 5.1

Given $f : (M, d) \to (N, \rho)$, the following are equivalent:

(i) f is continuous on M (in the ε-δ sense)

(ii) For any $x \in M$, if $x_n \to x$ in M, then $f(x_n) \to f(x)$ in N.

(iii) If E is closed in N, then $f^{-1}(E)$ is closed in M.

(iv) If V is open in N, then $f^{-1}(V)$ is open in M.

<u>Proof</u>

Start with (i) $\Rightarrow$ (ii). Let $x \in M$ and $(x_n) \subset M$ so that $x_n \to x$. Given $\varepsilon > 0$ there exists $\delta > 0$ so that

$$d(y, x) < \delta \implies \rho(f(y), f(x)) < \varepsilon.$$

Since $x_n \to x$ we can take N so that

$$n \geq N \quad \Rightarrow \quad d(x_n, x) < \delta$$

$$\Rightarrow \rho(f(x_n), f(x)) < \varepsilon.$$

This means precisely that

$$\lim_{n \to \infty} f(x_n) = f(x).$$

Next, for (ii) $\Rightarrow$ (iii), suppose E is closed in (N, ρ). Given $(x_n) \subset f^{-1}(E)$ so that

$x_n \to x$ in M, we need to show that $x \in f^{-1}(E)$. But $(x_n) \subset f^{-1}(E) \Longrightarrow$ $(f(x_n)) \subset E$, and by (ii) $f(x_n) \to f(x)$ in N. Moreover, since E is closed, we have $f(x) \in E$, and so $x \in f^{-1}(E)$. But this is what we needed.

The implication (iii) $\Longrightarrow$ (iv) (and likewise (iv) $\Longrightarrow$ (iii)) follows from the relation

$$f^{-1}(A^c) = f^{-1}(A)^c$$

(see Problem 5.1 for this relation). To see this, suppose V is open and consider $f^{-1}(V)$. We know that V^c is closed, so $f^{-1}(V^c)$ is closed (by (iii)). But

$$f^{-1}(V^c) = f^{-1}(V)^c \Rightarrow f^{-1}(V)^c \text{ is closed.}$$

This implies $f^{-1}(V)$ is open, which is what we wanted to show.

Finally, for (iv) $\Rightarrow$ (i) we fix $x \in M$ and let $\varepsilon > 0$ be given. The set $B_\varepsilon^\rho(f(x))$ is open in (N, ρ), so by (iv) the set $f^{-1}(B_\varepsilon^\rho(f(x)))$ is open in (M, d). This means there exists $\delta > 0$ so that

$$B_\delta^d(x) \subset f^{-1}(B_\varepsilon^\rho(f(x))),$$

and we have seen in our previous lecture that this is equivalent to our ε-δ definition of continuity. $\square$