Note Title 7/17/2015

Homeomorphisms

As mentioned in Problem 3.42 two metrics d and ρ on a set M are said to be equivalent if they generate the same convergent sequences: i.e., if $d(x_n, x) \longrightarrow 0$ iff $\rho(x_n, x) \longrightarrow 0$.

This idea relates two metric spaces (M, d) and (M, ρ), and in this lecture we would lik to expand on this idea by relating metric spaces based on different sets (M, d) and (N, ρ).

To get an idea of where we're headed, let's first consider $M = (0,1)$ and $N = (0,2)$, both with the usual (absolute value) metric. These sets are related by the function $f(x) = 2x$ $(f: M \to N)$, which is easily seen to be invertible, with inverse $f^{-1}(x) = \frac{x}{2}$. Such a function is an example of a homeomorphism.

Definition

Given two sets M and N, we say $f: M \to N$ is a homeomorphism if it is 1-1 and onto (and so invertible), continuous, and has a continuous inverse. If there exists such a map between two metric spaces (M, d) and (N, ρ), we say the spaces are homeomorphic.

Theorem 5.5

Let $f : (M, d) \to (N, \rho)$ be 1-1 and onto.

Then the following are equivalent:

(i) f is a homeomorphism

(ii) $x_n \xrightarrow{d} x \iff f(x_n) \xrightarrow{\rho} f(x)$

(iii) G is open in $M \iff f(G)$ is open in N.

(iv) E is closed in $M \iff f(E)$ is closed in N

(v) $\hat{d}(x,y) := \rho(f(x), f(y))$ defines a metric on M equivalent to d.

This will be proven in the homework, using the following lemma:

Lemma 5.7

Let $f : L \to M$ and $g : M \to N$, where L, M, and N are metric spaces (possibly with different metrics). If f is continuous at $x \in L$ and g is continuous at $f(x) \in M$, then $g \circ f : L \to N$ is continuous at $x \in L$. Here, $g \circ f(x) = g(f(x))$.

Proof

Let $x_n \to x$ in L, and notice that $f(x_n) \to f(x)$ in M (by the continuity of f) and $g(f(x_n)) \to g(f(x))$ by the continuity of g. But this say precisely that

$$x_n \to x \text{ in } L \Rightarrow g \circ f(x_n) \to g \circ f(x) \text{ in } N$$

and we conclude continuity. $\square$