Examples of Homeomorphisms

Note Title 7/17/2015

1. We've already seen that $(0,1)$ is homeomorphic to $(0,2)$. In fact any intervals with matching endpoint types are homeomorphic. For example, $(-3,1]$ is homeomorphic to $(0,1]$. The homeomorphism is the linear map between sets, and it's easiest to construct $f:(0,1] \to (-3,1]$: $f(x) = 4x - 3$.

We can also switch endpoint types, so $[-3, 1)$ is homeomorphic to $(0, 1]$. In this case, we need to map 0 to 1 and -3 to 0:

$$f : (0,1] \longrightarrow [-3, 1)$$

defined by $f(x) = -4x + 1$.

2. The interval $(0,1)$ is homeomorphic to $\mathbb{R}$. We can easily see this, because $f(x) = \tan x$ is a homeomorphism from $(-\pi/2, \pi/2)$ to $\mathbb{R}$, and

$g(x) = \pi x - \pi/2$ is a homeomorphism from $(0,1)$ to $(-\pi/2, \pi/2)$. So $f \circ g$ is a homeomorphism from $(0,1)$ to $\mathbb{R}$.

3. The interval $(0,1]$ is <u>not</u> homeomorphic to $(-3,1)$ (or any other interval with both endpoints open). To see this, suppose $(0,1]$ and $(-3,1)$ are homeomorphic with some homeomorphism f. We must have $f(1) \in (-3,1)$,

and we can denote this value c. (I.e., $c = f(1)$) Eliminating 1 from $(0,1]$ and c from $(-3,1)$ this generates a homeomorphism from $(0,1)$ to $(-3,c) \cup (c,1)$. But since $(0,1)$ is homeomorphic to $\mathbb{R}$, this sets up a homeomorphism from $(-3,c) \cup (c,1)$ to $\mathbb{R}$.

We have, then, a homeomorphism

$$f : (-3,c) \cup (c,1) \longrightarrow \mathbb{R}$$

and by Theorem 5.5 $f((-3,c))$ and $f((c,1))$ are open sets in $\mathbb{R}$. Since f is 1-1 these sets must be disjoint, and since f is onto we must have

$$f((-3,c)) \cup f((c,1)) = \mathbb{R}.$$

I.e., $\mathbb{R}$ is the union of two disjoint open sets. But this is a contradiction, because (as we'll see in Chapter 6) $\mathbb{R}$ is connected (i.e., cannot be expressed as the union of two non-trivial disjoint open sets).

4. For a curious example, let d denote the absolute value metric on $\mathbb{R}$, and ρ the discrete metric. Then $(\mathbb{R}, d)$ and $(\mathbb{R}, \rho)$ are not homeomorphic. This is clear from Theorem 5.5 and the fact that <u>all</u> subsets of $(\mathbb{R}, \rho)$ are both open and closed (and

this is certainly not the case for $(\mathbb{R}, d))$.