Connectedness

Note Title 7/19/2015

The set $(0,1)$ is clearly one piece, while the set $(0,1/2) \cup (1/2,1)$ is not, because there's a "hole" at $x = 1/2$. We will say $(0,1)$ is connected and $(0,1/2) \cup (\frac{1}{2},1)$ is disconnected.

As a starting point for this discussion, we'll show that an interval $[a,b]$ cannot be expressed as the union of two disjoint, non-empty, relatively open sets (open relative to $[a,b]$; recall that $[a,b]$ itself is open relative to $[a,b]$). Ultimately, we will express connectedness in terms of whether or not we can express a set in this way.

We'll argue by contradiction, assuming we can write $[a,b] = A \cup B$, where A and B are disjoint, non-empty, relatively open subsets of $[a,b]$. The point b must be in one of the sets, and let's assume $b \in B$. Since B is open this means $(b-\varepsilon, b] \subset B$ for some $\varepsilon > 0$ sufficiently small. Set $c = \sup A$, and notice that $c \leq b - \varepsilon$. Also, notice that A cannot only contain $\{a\}$ (because it's open), so we must $c > a$. $\Longrightarrow$ $a < c < b$.

We must have $(c-\varepsilon, c) \cap A \neq \emptyset$ and $(c, c+\varepsilon) \cap B \neq \emptyset$ for any $\varepsilon > 0$. By Proposition 4.10 this means $c \in \bar{A}$ and $c \in \bar{B}$, so $c \in \bar{A} \cap \bar{B}$. But $A = [a,b] \setminus B$ and $B = [a,b] \setminus A$, and since A and B are both open, we see that A and B are both closed. Then $c \in \bar{A} \cap \bar{B} = A \cap B$, and this is a contradiction to the assumption that A and B are disjoint.