Connectedness and Continuous Functions

Note Title 7/21/2015

Lemma 6.5

M is disconnected iff there exists a continuous map from M onto $\{0,1\}$ (the two-point discrete space).

Proof

For ($\Leftarrow$) suppose such a map exists. Since f is onto we have $A = f^{-1}(0)$ and $B = f^{-1}(1)$

satisfy:

$$A \cap B = \emptyset \quad \text{and} \quad A \cup B = M.$$

Also, the sets $\{0\}$ and $\{1\}$ are open in the discrete space, so A and B are open (as inverse images of open sets). This means A and B form a disconnection of M.

Conversely, for ($\Longrightarrow$) suppose M is disconnected so that there exists a disconnection A, B. Set

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B, \end{cases}$$

and note that f must be continuous because $f^{-1}(0) = A$, $f^{-1}(1) = B$, and $f^{-1}([0,1]) = M$, and these are all possible inverse images. $\square$

Lemma 6.5 leads to a generalization of the Intermediate Value Theorem: simply put, continuous functions map connected sets to connected sets.

Theorem 6.6

Let $f:(M,d)\to(N,\rho)$ be continuous, and let $E\subset M$. If E is connected then $f(E)$ is connected.

Proof

Suppose E is connected, but $f(E)$ is not. Then by Lemma 6.5 there exists an onto map $g : f(E) \to \{0,1\}$. But then $g \circ f : E \to \{0,1\}$ is continuous and onto, and this is a contradiction, because we would conclude that E is disconnected. $\square$