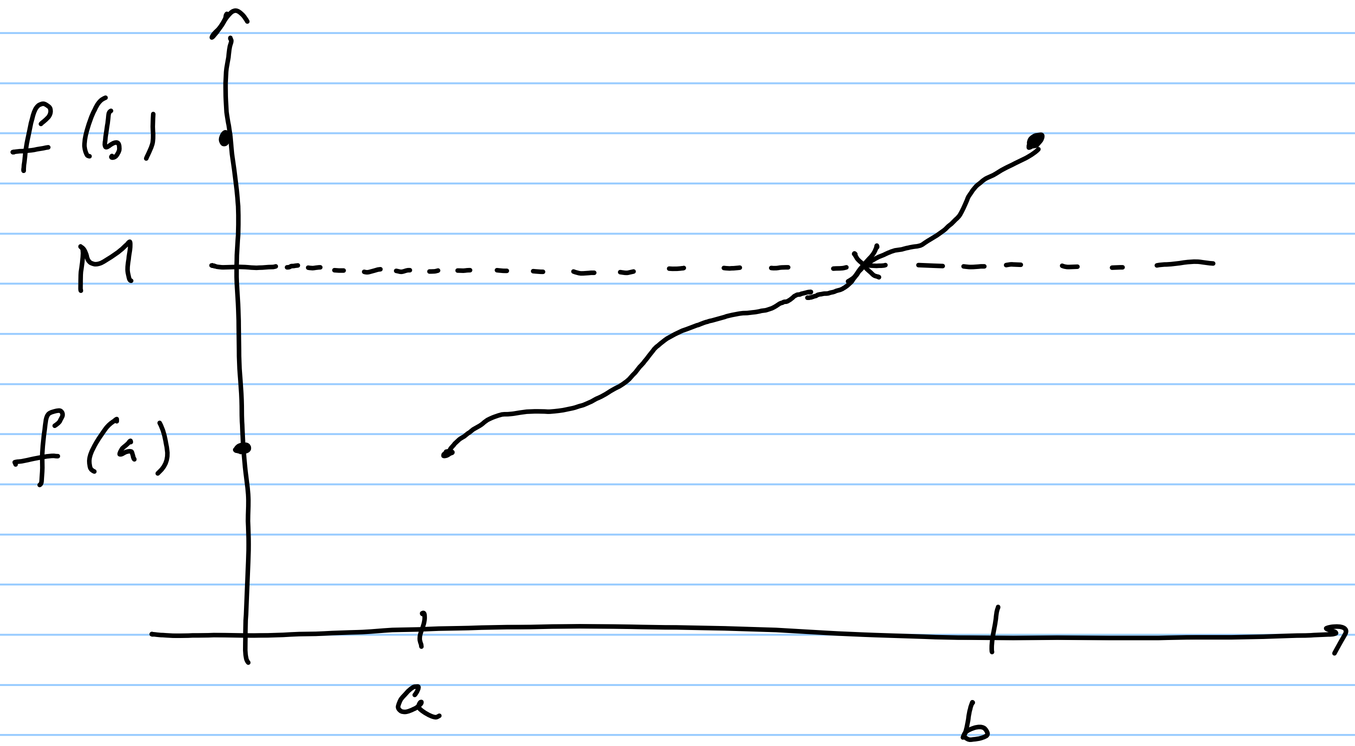
The Intermediate Value Theorem

Note Title 7/21/2015

Corollary 6.7 (Intermediate Value Theorem)

If I is an interval in $\mathbb{R}$ and if $f: I \to \mathbb{R}$ is a nonconstant continuous function, then $f(I)$ is an interval. In particular, if $a, b \in I$ with $f(a) \neq f(b)$ then f assumes every value between $f(a)$ and $f(b)$.

Standard picture from calculus



Proof

If I is an interval, then by Theorem 6.4 it is connected, and by Theorem 6.6 $f(I)$ is connected. But then again by Theorem 6.4 $f(I)$ is an interval. This gives the first statement of the theorem. The second statement is clear, because if an interval contains two points it must contain every point between those points (by definition of an interval). □

Example 6.8

We can use these ideas to clarify an example from Chapter 5. Recall that we said that no two intervals $[a,b]$, $(a,b]$, and (a,b) can be homeomorphic (where $(a,b]$ could be replaced by $[a,b)$). To see this in our new setting, suppose $f:(a,b] \to (a,b)$ is a homeomorphism, and let $c = f(b)$, noting that $a < c < b$. Then

$$f:(a,b) \to (a,c) \cup (c,b)$$

would be a homeomorphism. But this would be a contradiction, because a continuous function can't map a connected set into a disconnected set. The other cases are similar.