Totally Bounded Sets

Note Title 7/24/2015

Definition (totally bounded set)

We say that a subset A of a metric space (M,d) is *totally bounded* if given any $\varepsilon > 0$ there exist finitely many points $\{x_i\}_{i=1}^{n} \subset M$ so that

$$A \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i).$$

It's fair to ask: are there metric spaces for which total boundedness is different from boundedness? Yes, and here's an example:

Consider ℓ_1, which we recall is the space of all infinite sequences $a = (a_n)$ so that

$$\|a\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty .$$

The (standard) unit vectors in this space are

$$\hat{e}_i = (0, \dots, 0, 1, 0, \dots, 0, \dots)$$

$\hat{}$ i^{th} slot

Since $\|\hat{e}_i\|_1 = 1 \quad \forall i$, the set

$$A = \{ \hat{e}_i : i = 1, 2, \dots \}$$

is bounded. However, for $i \neq j$

$$\|\hat{e}_i - \hat{e}_j\|_1 = 2,$$

so if we take $\varepsilon < 2$ we must cover each $\hat{e}_i$ with a different ball, and this will require an infinite number of balls.

Returning to our definition of totally bounded sets, the condition

$$A \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i)$$

is sometimes characterized by saying that the set $\{x_1, x_2, \ldots, x_n\}$ is "ε-dense in A" or is "an ε-net for A." More commonly, it's said (as Carothers states) that A is <u>covered</u>

by the set

$$\mathcal{B} = \{ B_\varepsilon(x_i) : i = 1, 2, \dots, n \}.$$