Complete Metric Spaces

Note Title 7/25/2015

Definition

A metric space (M,d) is said to be <u>complete</u> if every Cauchy sequence in (M,d) converges to a point in M.

We've either seen or will see that each of the following is complete with the usual metric: $\mathbb{R}$, $\mathbb{R}^n$, c_0, ℓ_1, ℓ_2, ℓ_∞, $C[a,b]$.

When working with the space c_0, ℓ_1, ℓ_2, and ℓ_∞, it will be convenient to use function notation. For example, a sequence $(a_k) \in \ell_2$ will be expressed as $f = (f(k))_{k=1}^{\infty}$, so $f \in \ell_2$, and $f(k) = a_k$, $k = 1, 2, \ldots$

This will allow us to express a sequence of sequences $(f_n) \subset \ell_2$, where each element f_n is a sequence $f_n = (f_n(k))_{k=1}^{\infty}$.

As an example, with this notation the standard basis elements in l_2 can be expressed as

$$e_n = (e_n(k))_{k=1}^{\infty},$$

where

$$e_n(k) = \delta_{n,k} = \begin{cases} 1 & n=k \\ 0 & n \neq k \end{cases}$$

This object $\delta_{n,k}$ is often referred to as Kronecker's delta.