Note Title ℓ_2 is Complete 7/26/2015

Theorem

ℓ_2 is complete with the metric

$$d(x,y) = \|x-y\|_2 .$$

Proof

Let (f_n) be a Cauchy sequence in ℓ_2. We need to show that (f_n) converges to $f \in \ell_2$. First, observe that for each fixed index k the values

$(f_n(k))_{n=1}^{\infty}$ (i.e., the sequences of k^{th} components) form a Cauchy sequence in $\mathbb{R}$, and so converge in $\mathbb{R}$. This provides us with a candidate for our limit f. Precisely, for $k = 1, 2, \ldots$ we set

$$f(k) := \lim_{n \to \infty} f_n(k)$$

This defines a sequence $(f(k))$, but we need to check two things: (1) that this sequence is in l_2; and (2) that $\{f_n\}$ converges to this sequence in the l_2 norm.

For (1), we've seen that Cauchy sequences are bounded, so there exists some constant C so that

$$\|f_n\|_2 \leq C \quad \forall\, n = 1, 2, \ldots$$

If we fix an integer N then

$$\sum_{k=1}^{N} |f(k)|^2 = \lim_{n \to \infty} \sum_{k=1}^{N} |f_n(k)|^2$$

But

$$\lim_{n\to\infty} \sum_{k=1}^{N} |f_n(k)|^2 \leq \lim_{n\to\infty} \|f_n\|_2^2 \leq C^2.$$

We see that

$$\sum_{k=1}^{N} |f(k)|^2 \leq C^2,$$

and since this is true for all N we can take $N \to \infty$ to see that

$$\|f\|_2^2 \leq C^2 \Rightarrow f \in \ell_2.$$

Finally, for (2), let $\varepsilon > 0$ be given and take (by the Cauchy property) $\tilde{N}$ large enough so that

$$m, n \geq \tilde{N} \implies \|f_n - f_m\|_2 < \varepsilon.$$

Then for any N and $n \geq \tilde{N}$

$$\sum_{k=1}^{N} |f(k) - f_n(k)|^2 = \lim_{m \to \infty} \sum_{k=1}^{N} |f_m(k) - f_n(k)|^2$$

$$\leq \lim_{m \to \infty} \|f_m - f_n\|_2^2 \leq \varepsilon^2.$$

This gives

$$\sum_{k=1}^{N} |f(k) - f_n(k)|^2 \leq \varepsilon^2.$$

Similarly as with (1), since the right-hand side is independent of N we can take $N \to \infty$ to see that

$$\|f - f_n\|_2^2 \leq \varepsilon^2$$

$\Rightarrow \|f - f_n\|_2 \leq \varepsilon$, and since ε is arbitrary

We conclude that $f_n \to f$ in ℓ_2. $\square$