Characterizations of Complete Metric Spaces, I

Note Title 7/26/2015

Theorem 7.11

For any metric space (M,d) the following statements are equivalent:

(i) (M,d) is complete

(ii) Let $F_1 \supset F_2 \supset F_3 \supset \cdots$ be a decreasing sequence of nonempty closed sets in M with $\operatorname{diam}(F_n) \to 0$. Then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

(iii) Every infinite, totally bounded subset of M has a limit point in M.

Proof that (i) $\Rightarrow$ (ii)

Given a sequence (F_n) as described in (ii), for each n choose $x_n \in F_n$. Since (F_n) is decreasing (by containment)

$$\{x_k : k \geq n\} \subset F_n,$$

and since $\operatorname{diam}(F_n) \to 0$ this means (x_n) is a Cauchy sequence. That is, by taking n large, we can make $\operatorname{diam}\{x_k : k \geq n\}$ as small as we like, which means $d(x_k, x_j)$ are as small as we like for $j, k \geq n$. By (i) (x_n) converges to $x \in M$. Since $\{x_k : k \geq n\} \subset F_n$ we can view this part of the sequence as a sequence in F_n, and since F_n is closed it converges to $x \in F_n$. This is true for all n, so $x \in F_n$ for all n, which implies

$$x \in \bigcap_{n=1}^{\infty} F_n.$$

This shows that $\bigcap_{n=1}^{\infty} F_n$ contains at least one point. But if it contains a second point, say y, we must have $d(x,y) > 0$, and this contradicts $\mathrm{diam}(F_n) \to 0$ as $n \to \infty$.