Characterizations of Complete Metric Spaces, II

Note Title 7/26/2015

Theorem 7.11

For any metric space (M,d) the following statements are equivalent:

(i) (M,d) is complete

(ii) Let $F_1 \supset F_2 \supset F_3 \supset \cdots$ be a decreasing sequence of nonempty closed sets in M with $\operatorname{diam}(F_n) \to 0$. Then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

(iii) Every infinite, totally bounded subset of M has a limit point in M.

<u>Proof that (ii) $\Rightarrow$ (iii)</u>

Let A be an infinite, totally bounded subset of M. We showed in the proof of Part (ii) of Lemma 7.3 that A contains a sequence (x_n) (a subsequence in that proof) that is Cauchy and comprised of distinct points

$x_n \neq x_m$ for $n \neq m$. Set

$$A_n := \{x_k : k \geq n\}$$

and notice that

$$A_1 \supset A_2 \supset \cdots$$

By the Cauchy property $\operatorname{diam}(A_n) \longrightarrow 0$ as $n \to \infty$. The only thing missing in the application of (ii) is that the A_n are not closed, but we can work with closures $\overline{A_n}$.

For these closures, we have

$$\bar{A}_1 \supset \bar{A}_2 \supset \dots$$

and $\operatorname{diam}(\bar{A}_n) = \operatorname{diam}(A_n)$ (by the definition of diameter). Now we can apply (ii) to get the existence of

$$x = \bigcap_{n=1}^{\infty} \bar{A}_n .$$

Since $x \in \bar{A}_n$ $\forall n$ we must have

$$\operatorname{dist}(x_n, x) \leq \operatorname{diam}(A_n) \longrightarrow 0$$

We see that $x_n \to x$, and so x is a limit point of A, and so of M.

For (iii) $\Rightarrow$ (i) we assume (iii) holds and let (x_n) denote a Cauchy sequence in (M, d). Since Cauchy sequences with convergent subsequences converge, we only need to show that (x_n) has a convergent subsequence. We know by (i) of Lemma 7.3 that $A = \{x_n : n \geq 1\}$ is totally bounded. If A contains only a finite number of distinct elements then we are done as usual by repeats, and if A has an infinite number of distinct points then (iii) implies that A has a limit point $x \in M$. But this means precisely that (x_n) has a convergent subsequence, and this is what we needed to show. $\square$