M642 Assignment 8, due Friday April 5

Note. This is a double assignment, since there is no class Friday March 29.

1. [10 pts] (**Keener Problem 6.1.1.**) Solve the following.

a. The function $f(z) = \sqrt{z(z-1)(z-4)}$ is defined to have branch cuts given by $z = \rho e^{-i\pi/2}$, $z = 1 + \rho e^{i\pi/2}$, and $z = 4 + \rho e^{-i\pi/2}$, $0 \leq \rho < \infty$, and with $f(2) = -2i$. Evaluate $f(-3)$, $f(1/2)$, and $f(5)$.

b. Determine the branch cut structure of

$$f(z) = \ln\Big(5 + \sqrt{\frac{z+1}{z-1}}\Big).$$

Make a plot in the f complex plane of the different branches of this function.

2. [10 pts] Consider the function

$$f(z) = \begin{cases} \frac{z^5}{|z|^4} & z \neq 0 \\ 0 & z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations hold at $z = 0$, but that f is not differentiable at $z = 0$.

3. [10 pts] (**Keener Problem 6.1.3.**) Find all values of z for which

(a) $\sin z = 2$

(b) $\sin z = i$

(c) $\tan^2 z = -1$

4. [10 pts] Let $r > 0$ and let $\gamma(t) = re^{it}$ for $t \in [0, \frac{\pi}{4}]$. Show that

$$\Big| \int_\gamma e^{iz^2} dz \Big| \leq \frac{\pi(1 - e^{-r^2})}{4r}.$$

Note. Although we may not have time to prove it in class, this problem assumes students are familiar with the inequality

$$\Big| \int_C f(z) dz \Big| \leq \int_C |f(z)||dz|.$$

5. [10 pts] In M641 we reviewed the integral formula

$$\int_U u_{x_i} d\vec{x} = \int_{\partial U} u\nu^i dS,$$

where $U \subset \mathbb{R}^n$ $(n = 1, 2, \dots)$ is an open set with C^1 boundary (piecewise C^1 is okay for $n = 2$) and ν^i denotes the i^{th} component of the outward unit normal vector. Use this to prove Green's Theorem

$$\int_C P dx + Q dy = \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx dy,$$

under the assumptions stated in class.

6. [10 pts] (**Keener Problem 6.2.1.**) The function $f(z)$ is analytic in the entire z plane including $z = \infty$ except at $z = i/2$, where it has a simple pole, and at $z = 2$ where it has a pole of order 2. It is known that

$$\int_{|z|=1} f(z)dz = 2\pi i$$
$$\int_{|z|=3} f(z)dz = 0$$
$$\int_{|z|=3} f(z)(z-2)dz = 0.$$

Find $f(z)$ (unique up to an arbitrary additive constant).

7. [10 pts] Suppose $f(z) = u + iv$ is analytic in an open set U.

a. Show that if $f'(z) = 0$ for all $z \in U$ then f is constant in U.

b. Show that if one of the functions u, v, or $|f|$ is constant in U then f is constant in U.

8. [10 pts] Evaluate each of the following integrals.

a. (**Keener Problem 6.2.6.**)

$$\int_{|z|=1/2} \frac{z+1}{z^2+z+1} dz.$$

b. (**Keener Problem 6.2.7.**)

$$\int_{|z|=1/2} e^{z^2 \ln(1+z)} dz.$$

c. (**Keener Problem 6.2.8.**)

$$\int_{|z|=1/2} \arcsin z dz.$$

d. (**Keener Problem 6.2.9.**)

$$\int_{|z|=1} \frac{\sin z}{2z+i} dz.$$

e. (**Keener Problem 6.2.10.**)

$$\int_{|z|=1} \frac{\ln(z+2)}{z+2} dz.$$

f. (**Keener Problem 6.2.11.**)

$$\int_{|z|=1} \cot z dz.$$

9. [10 pts] (**Keener Problem 6.2.12.**) Show that, if $f(z)$ is analytic and nonzero on the interior of some region, then $|f(z)|$ cannot attain a (strict) local minimum on the interior of the domain.

10. [10 pts] In this problem we'll define the concept of a winding number and determine one of its properties.

Definition. Suppose $\gamma : [a, b] \to \mathbb{C}$ is a closed, piecewise smooth path and that z is any point not on γ. The winding number $n(\gamma, z)$ of γ about z (also called the index of γ with respect to z) is defined as

$$n(\gamma, z) := \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}.$$

Show that $n(\gamma, z)$ is always integer-valued.

Note. Although we won't prove it, the following characterization can be made precise: $n(\gamma, z)$ counts the number of times γ winds around z. If γ is such that the bounded region surrounded by γ is kept to the left, then $n(\gamma, z)$ is positive.