Analysis of ODE Models

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Contents

1 Overview

In these notes we consider the three main aspects of ODE analysis: (1) existence theory, (2) uniqueness theory, and (3) stability theory. In our study of existence theory we ask the most basic question: are the ODE we write down guaranteed to have solutions? Since in practice we solve most ODE numerically, it's possible that if there is no theoretical solution the numerical values given will have no genuine relation with the physical system we want to analyze. In our study of uniqueness theory we assume a solution exists and ask if it is the only possible solution. If we are solving an ODE numerically we will only get one solution, and if solutions are not unique it may not be the physically interesting solution. While it's standard in advanced ODE courses to study existence and uniqueness first and stability later, we will start with stability in these note, because it's the one most directly linked with the modeling process. The question we ask in stability theory is, do small changes in the ODE conditions (initial conditions, boundary conditions, parameter values etc.) lead to large changes in the solution? Generally speaking, if small changes in the ODE conditions lead only to small changes in the solution we say the solution is stable.

2 Stability

In order to gain some insight into what stability is, we will consider the equation of a simple undamped pendulum,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta; \quad \theta(0) = \theta_0; \quad \theta'(0) = \omega_0, \tag{2.1}$$

where l denotes the length of the pendulum, g denotes the acceleration due to gravity at the earth's surface, and $\theta(t)$ is the pendulum's angle of displacement from the vertical. (A derivation of this equation is given in [H.2].) In order to solve equation (2.1) with MATLAB, we must first write it as a first order system. Taking $y_1 = \theta$ and $y_2 = \frac{d\theta}{dt}$, we have

$$\begin{aligned} \frac{dy_1}{dt} &= y_2; \quad y_1(0) = \theta_0, \\ \frac{dy_2}{dt} &= -\frac{g}{l}\sin y_1; \quad y_2(0) = \omega_0. \end{aligned} \tag{2.2}$$

Taking $l = 1$ and $g = 9.81$ we will store this equation in the MATLAB M-file *pendode.m,*

```
function yprime = pendode(t,y);
%PENDODE: Holds ODE for pendulum equation.
g = 9.81; l = 1;
yprime = [y(2);-(g/l)*sin(y(1))];
```

and solve it with the M-file *pend.m,*

```
function f = pend(theta0,v0);
%PEND: Solves and plots ODE for pendulum equation
%Inputs are initial angle and initial angular velocity
y0 = [theta0 v0];
tspan = [0 5];
[t,y] = ode45(@pendode,tspan,y0);
plot(y(:,1),y(:,2));
```

(These files are both available on the web site [H.1].) Taking initial angle $\pi/4$ and initial velocity 0 with the command *pend(pi/4,0)*, leads to Figure 2.1 (I've added the labels from MATLAB's pop-up graphics window).

Notice that time has been suppressed and the two dependent variables y_1 and y_2 have been plotted in what we refer to as a *phase portrait.* Beginning at the initial point $\theta_0 = \frac{\pi}{4}$,

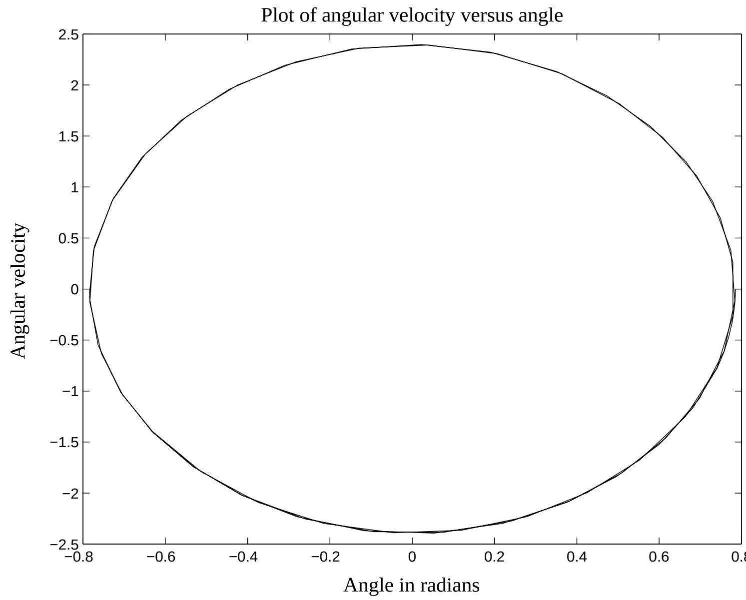

Figure 2.1: Pendulum motion for the case $\theta_0 = \frac{\pi}{4}$ and $\omega_0 = 0$.

$\omega_0 = 0$ (the right-hand tip of the football), we observe that angular velocity becomes negative (the pendulum swings to the left) and angle decreases. At the bottom of the arc, the angle is 0 but the angular velocity is at a maximum magnitude (though negatively directed), while at the left-hand tip of the football the object has stopped swinging (instantaneously), and is turning around. The remainder of the curve corresponds with the object's swinging back to its starting position. In the (assumed) absence of air resistance or other forces, the object continues to swing like this indefinitely. Alternatively, taking initial angle 0 and initial velocity 10 with the command *pend(0,10)* leads to Figure 2.2. Observe that in this case angular velocity is always positive, indicating that the pendulum is always swinging in the same (angular) direction: we have started it with such a large initial velocity that it's looping its axis.

Now that we have a fairly good idea of how to understand the pendulum phase diagrams, we turn to the critical case in which the pendulum starts pointed vertically upward from its axis (remember that we have assumed it is attached to a rigid rod). After changing the variable *tspan* in *pend* to $[0, 20]$ (solving now for 20 seconds), the command *pend(pi,0)* leads to Figure 2.3. In the absence of any force other than gravity, we expect our model to predict that the pendulum remains standing vertically upward. (What could possibly cause it to fall one way rather than the other?) What we find, however, is that our model predicts that it will fall to the left and then begin swinging around its axis.

Consider finally a change in this last initial data of one over one trillion ($10^{-12} = .000000000001$). The MATLAB command *pend(pi+1e-12,0)* produces Figure 2.4. We see that with a change in initial data as small as 10^{-12} radians, the change in behavior is enormous: the pendulum spins in the opposite direction. We conclude that our model, at least as it is solved on MATLAB, fails at the initial data point $(\pi, 0)$. In particular, we say that this point is unstable.

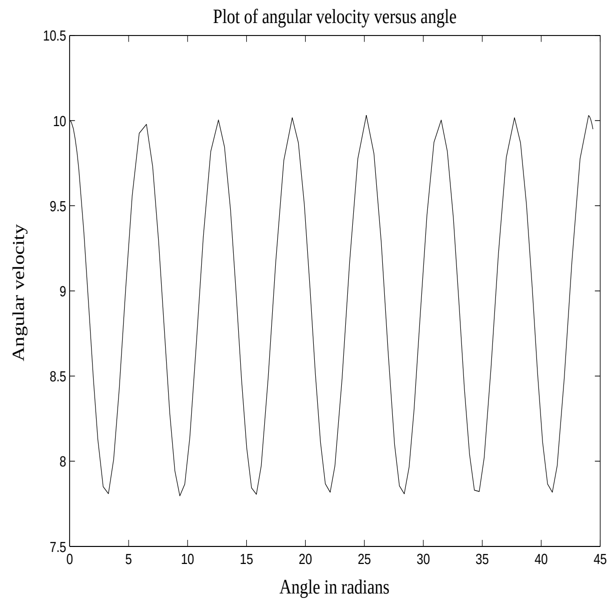

Figure 2.2: Pendulum motion for the case $\theta_0 = 0$ and $\omega_0 = 10$ s^{-1}.

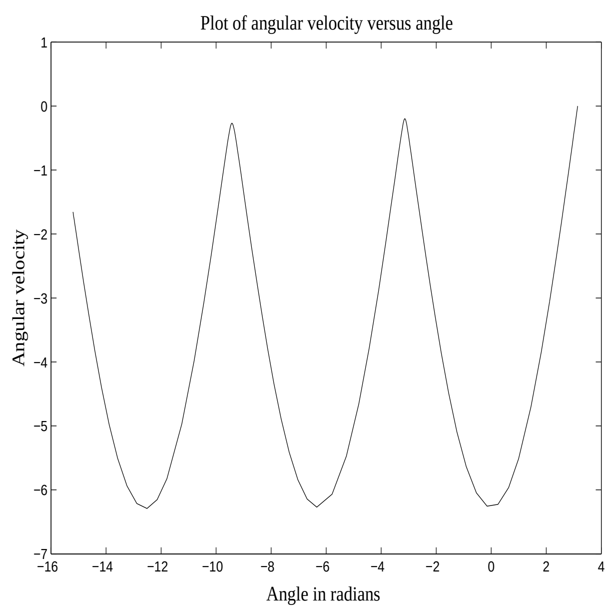

Figure 2.3: Pendulum motion for the case $\theta_0 = \pi$ and $\omega_0 = 0$ s^{-1}.

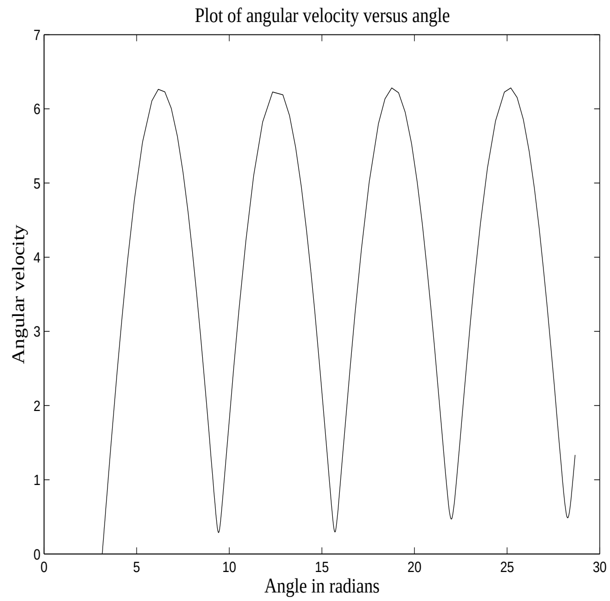

Figure 2.4: Pendulum motion for the case $\theta_0 = \pi + 10^{-12}$ and $\omega_0 = 0$ s^{-1}.

In what follows we will carry out a systematic development of some of the main ideas in stability theory.

2.1 The Phase Plane

Example 2.1. For systems of two first-order differential equations such as (2.2), we can study phase diagrams through the useful trick of dividing one equation by the other. We write,

$$\frac{dy_2}{dy_1} = \frac{\frac{dy_2}{dt}}{\frac{dy_1}{dt}} = \frac{-\frac{g}{l}\sin y_1}{y_2},$$

(the *phase-plane equation*) which can readily be solved by the method of separation of variables for solution

$$\frac{y_2^2}{2} = \frac{g}{l}\cos y_1 + C. \tag{2.3}$$

We see that each value of the constant C gives a relationship between y_1 and y_2, which we refer to as an *integral curve*. At $t = 0$, $x_1(0) = \theta_0$ and $x_2(0) = \omega_0$, fixing C. We will create a phase plane diagram with the M-file *penphase.m* (available at [H.1]):

```
function f = penphase(theta,w0);
%PENPHASE: Plots phase diagram for
%pendulum equation with initial angle theta
%and initial angular velocity w0.
g = 9.81; l = 1.0;
C = w0^2/2 - (g/l)*cos(theta);
if C > g/l
```

```
y = linspace(-pi,pi,50);
else
maxtheta = acos(-C*l/g); %Maximum value of theta
y = linspace(-maxtheta,maxtheta,50);
end
up = sqrt(2*g/l*cos(y)+2*C);
down = -sqrt(2*g/l*cos(y)+2*C);
plot(y,up);
hold on
plot(y,down);
```

Typing in sequence *penphase(pi/12,0), penphase(pi/4,0), penphase(pi/2,0), penphase(pi,0), penphase(pi/4,6)*, we create the phase plane diagram given in Figure 2.5. Though time has been suppressed here, we can determine the direction in which solutions move along these integral curves by returning to the original system. From the equation

$$\frac{dy_1}{dt} = y_2,$$

we see that for $y_2 > 0$ values of $y_1(t)$ are increasing, and so the direction is to the right. On the other hand, for $y_2 < 0$, values of $y_1(t)$ are decreasing, and so the direction is to the left. We can conclude that as time progresses solutions move clockwise about $(0,0)$.

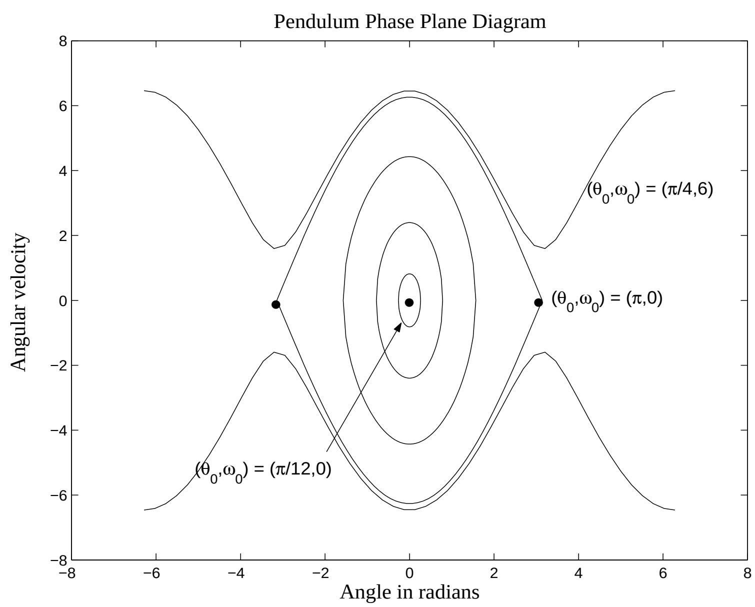

Figure 2.5: Phase plane diagram for a simple pendulum (Example 3.1).

The point $(\theta_0, \omega_0) = (0,0)$ corresponds with the pendulum's hanging straight down, while the points $(\theta_0, \omega_0) = (\pi, 0)$ and $(\theta_0, \omega_0) = (-\pi, 0)$ both correspond with the pendulum's standing straight up above its axis. Notice that at each of these *critical* or *equilibrium* points our model analytically predicts that the pendulum will not move. For example, at

$(\theta_0, \omega_0) = (0,0)$ we find from (2.2) that $\frac{dy_1}{dt} = \frac{dy_2}{dt} = 0$: the angle and angular velocity are both zero, so the pendulum remains at rest. The curve that connects the points $(-\pi, 0)$ and $(\pi, 0)$ separates the phase plane into its two parts: the trajectories inside this curve correspond with the pendulum's swinging back and forth, while the trajectories outside this curve correspond with the pendulum's swinging entirely around its axis. We refer to this curve as a *separatrix* for the system. △

Definition 2.1. (Equilibrium point) For an autonomous system of ordinary differential equations

$$\frac{d\vec{y}}{dt} = \vec{f}(\vec{y}), \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \vec{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}, \tag{2.4}$$

we refer to any point $\vec{y}_e$ so that $\vec{f}(\vec{y}_e) = \vec{0}$ as an *equilibrium point.*

Typically, equilibrium points govern long time behavior of physical models. In particular, solutions often approach (or remain near) stable equilibrium points as time gets large.

Returning to our pendulum analysis, we note that if we perturb the initial point $(0,0)$ a little (by pushing the pendulum slightly to the right or left), the pendulum's behavior changes only slightly: if we push it one millimeter to the right, it will swing back and forth with maximum displacement one millimeter. On the other hand, as we have seen, if we perturb the initial point $(\pi, 0)$ the pendulum's behavior changes dramatically. We say that $(0,0)$ is a stable equilibrium point and that $(\pi, 0)$ and $(-\pi, 0)$ are both unstable equilibrium points. More precisely, we say that the point $(0,0)$ is *orbitally stable*, which signifies that solutions don't actually approach it, but rather remain near it.

Definition 2.2. (Stability) *An equilibrium point $\vec{y}_e$ for (2.4) is called* stable *if given any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ so that*

$$|\vec{y}(0) - \vec{y}_e| < \delta(\epsilon) \Rightarrow |\vec{y}(t) - \vec{y}_e| < \epsilon \quad \textit{for all } t > 0.$$

Definition 2.3. (Asymptotic stability.) *An equilibrium point $\vec{y}_e$ for (2.4) is called* asymptotically stable *if it is stable and additionally there exists some $\delta_0 > 0$ so that*

$$|\vec{y}(0) - \vec{y}_e| < \delta_0 \Rightarrow \lim_{t \to \infty} \vec{y}(t) = \vec{y}_e.$$

(These definitions can be extended to the case of non-autonmous systems, and to more general solutions such as periodic solutions, but we won't pursue that here.)

Before leaving our pendulum example, we consider a rigorous procedure for establishing the existence of orbits (closed curves) in the phase plane. (I.e., a method that won't require MATLAB.) Recalling that our pendulum equation, in its original form, is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta,$$

we consider the class of second order ODE with the form

$$y'' + V'(y) = 0. \tag{2.5}$$

For example, the pendulum equation has this form with $y = \theta$ and $V(y) = -\frac{g}{l}\cos\theta$. Proceeding as we did with the pendulum equation, we can write $y_1 = y$ and $y_2 = y'$, so that (2.5) becomes the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -V'(y_1), \end{aligned}$$

with the associated phase plane equation

$$\frac{dy_2}{dy_1} = -\frac{V'(y_1)}{y_2}.$$

Separating variables and integrating, we find

$$\frac{y_2^2}{2} + V(y_1) = E_0 = \text{constant}, \tag{2.6}$$

where the constant has been denoted by E_0 in recognition of the fact that the left hand side of (2.6) has the general form of kinetic energy plus potential energy, and indeed this is what the terms correspond with in many applications. If we now return to the y variable, we have the first order equation

$$\frac{dy}{dt} = \pm\sqrt{2(E_0 - V(y))},$$

for which the right hand side is defined so long as $E_0 \geq V(y)$. Recalling that the phase plane in this case consists of a plot of $\frac{dy}{dt}$ versus y, we see that there is a curve in the phase plane at the values y for which $E_0 \geq V(y)$, and that by symmetry the curve will look exactly the same for $\frac{dy}{dt} > 0$ as it does for $\frac{dy}{dt} < 0$. This situation is depicted graphically in Figure 2.6, where for this choice of $V(y)$ there is no curve between y_1 and y_2, and the two symmetric curves between y_2 and y_3 form a closed curve. Note that the closed curve corresponds with the constant E_0, and as with our pendulum example, we get a different integral curve for each value of E_0.

Example 2.1 cont. As a final remark on Example 3.1, we will apply the above considerations to the pendulum equation. In this case $V(y) = -(g/l)\cos(y)$, which has a local minimum at $-g/l$, as depicted in Figure 2.7. (Keep in mind that we know from Example 2.10 that the potential energy for the pendulum is actually $P(y) = mgl(1 - \cos y)$, which is always positive.) We see, then, that for $E_0 < -g/l$ there is no curve in the phase plane, while for each E_0 satisfying $-g/l < E_0 < g/l$ there is precisely one closed orbit. For $E_0 > g/l$ there are two phase plane curves, each defined for all values of y, one that is always positive and one that is always negative. △

Example 2.2. Find all equilibrium points for the SIR model

$$\begin{aligned} \frac{dS}{dt} &= -aSI; \quad S(0) = S_0 \\ \frac{dI}{dt} &= aSI - bI, \quad I(0) = I_0, \end{aligned}$$

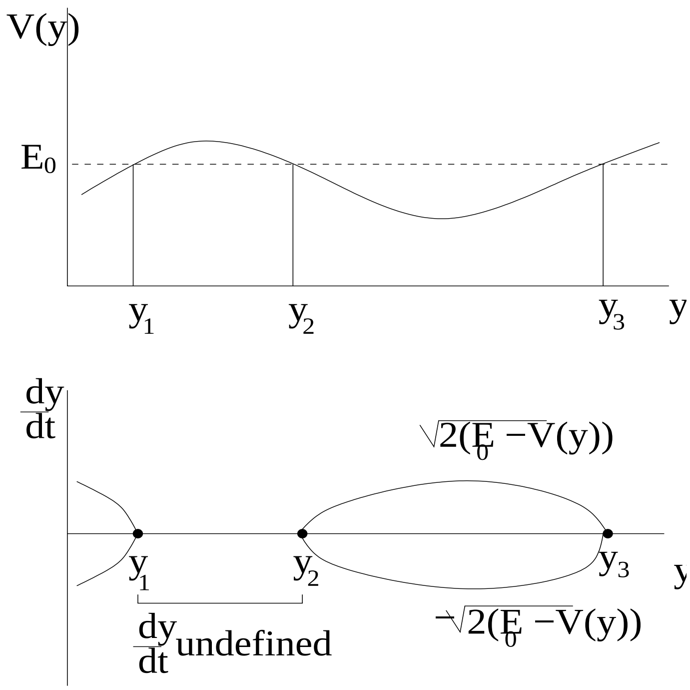

Figure 2.6: Plot of a generic potential function $V(y)$ and an associated periodic orbit.

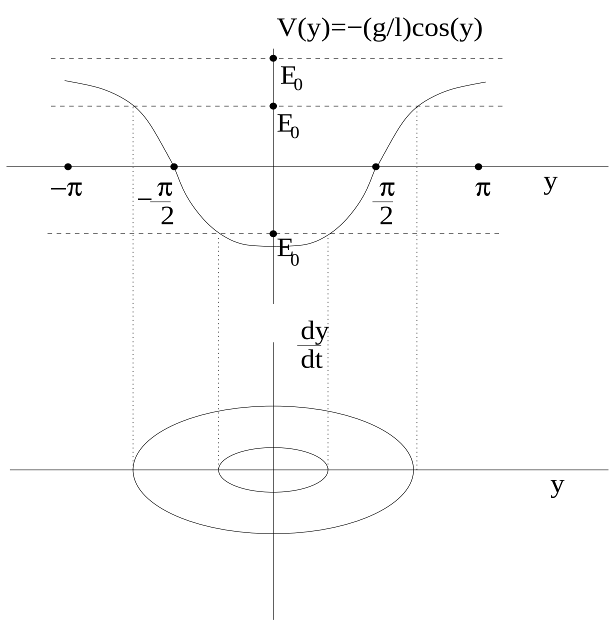

Figure 2.7: Plot of the potential function for the pendulum equation and the phase plane.

and use a phase plane analysis to determine whether or not each is stable. (Observe that since the variable R does not appear in either of these equations, we are justified in dropping off the equation for R. We recover R from the relation $R = N - S - I$, where N is the total number of individuals in the population.)

We begin by searching for equilibrium points (S_e, I_e), which satisfy

$$\begin{aligned} aS_eI_e &= 0 \\ aS_eI_e - bI_e &= 0. \end{aligned}$$

From these equations, we can conclude $I_e = 0$ and S_e can be any number, though since it represents a population we will keep it positive. The phase plane equation is

$$\frac{dI}{dS} = -\frac{aSI - bI}{aSI} = -1 + \frac{b}{aS},$$

which can be solved by separation of variables. We find

$$I(S) = -S + \frac{b}{a}\ln S + C.$$

It's clear from the phase plane equation that the only critical point for $I(S)$ is $S_c = \frac{b}{a}$, and that this corresponds with a maximum value. In addition to this, for any values of C, we have the limits

$$\lim_{S\to 0^+} I(S) = -\infty; \quad \lim_{S\to\infty} I(S) = -\infty,$$

which shows that if $I(b/a) > 0$ then there will be values $S_R > b/a$ and $S_L < b/a$ so that $I(S_R) = I(S_L) = 0$. (If $I(b/a) \le 0$ there is no infection.) Finally, it's clear from the first equation $S' = -aSI$ that as long as the populations S and I are both positive, $S' < 0$, which means arrows on our trajectories move to the left. Combining these observations, we have Figure 2.8. Suppose now that the initial values (S_0, I_0) are near an equilibrium point $(S_e, 0)$ for which $S_e > \frac{b}{a}$. In this case the solution will follow a trajectory up and to the left, moving away from the initial value. We conclude that all such equilibrium points are unstable. What this means physically is that if the susceptible population is $S_e > \frac{b}{a}$, then if a single person becomes infected the number of infected people will begin to grow. On the other hand, if the initial values (S_0, I_0) are near an equilibrium point $(S_e, 0)$ for which $S_e < \frac{b}{a}$, we see that the solution will follow a trajectory that returns *to a nearby equilibrium point* $(S_n, 0)$. We conclude that these equilibrium points are stable, though not asymptotically stable. Physically, this means that if the susceptible population is $S_e < \frac{b}{a}$, then if a single person is infected, or if a small number of people are infected, the infection will not spread. Technically, the equilibrium point $(\frac{b}{a}, 0)$ is unstable, but only because mathematically we can think of negative-population perturbations. Note also that though points just to the right of $(\frac{b}{a}, 0)$ are technically unstable, they do not correspond with physically unstable behavior. That is, if we are to the right of $\frac{b}{a}$, but quite close to it, the number of infected people will only increase a little before dropping off again. $\triangle$

2.2 Linearization

In this section, we will consider a method for greatly simplifying the stability analysis for nonlinear systems of ODE. It should be stressed, however, at the outset that the method introduced in this section is not rigorous without further work. (See Subsection 2.9.)

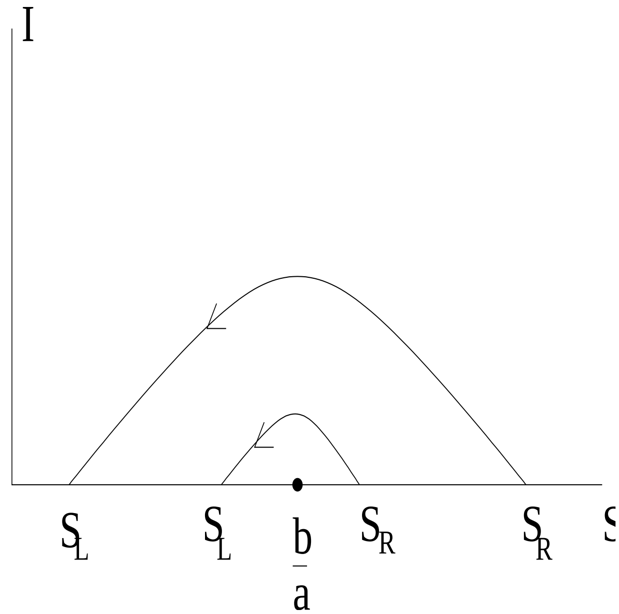

Figure 2.8: Phase plane diagram for the SIR model.

Example 2.1 cont. We return once again to the pendulum equation from Example 3.1. In general, we can study stability without solving equations quite as complicated as (2.3). Suppose we want to analyze stability at the point $(0, 0)$. We first recall the Taylor expansion of $\sin y$ near $y = 0$,

$$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} + ...$$

For y near 0, higher powers of y are dominated by y, and we can take the (small angle) approximation, $\sin y \cong y$, which leads to the *linearized* equations,

$$\begin{aligned} \frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= -\frac{g}{l} y_1. \end{aligned} \tag{2.7}$$

(That is, the right-hand sides of (2.7) are both linear, which will always be the case when we take the linear terms from a Taylor expansion about an equilibrium point.) Of course it is precisely here, in dropping off these nonlinear terms, that we have lost rigour, and to have a complete theory we will have to say more about them later. For now, though, we will consider what can be said about the linear system. The phase plane equation associated with (2.7) is

$$\frac{dy_2}{dy_1} = \frac{\frac{dy_2}{dt}}{\frac{dy_1}{dt}} = \frac{-\frac{g}{l} y_1}{y_2},$$

with solution

$$\frac{y_2^2}{2} + \frac{g}{l} \cdot \frac{y_1^2}{2} = C,$$

which corresponds with ellipses centered at $(0,0)$ with radial axis lengths $\sqrt{2C}$ and $\sqrt{2lC/g}$ (see Figure 2.9). Returning to equations (2.7), we add direction along the ellipses by observing from the first equation that for $y_2 > 0$, $y_1$is increasing, and for $y_2 < 0$, y_1 is decreasing. The directed sections of integral curves along which the object moves are called *trajectories*. Our stability conclusion is exactly the same as we drew from the more complicated Figure 2.5. In particular, in the case that we have closed loops about an equilibrium point, we say the point is *orbitally stable.*

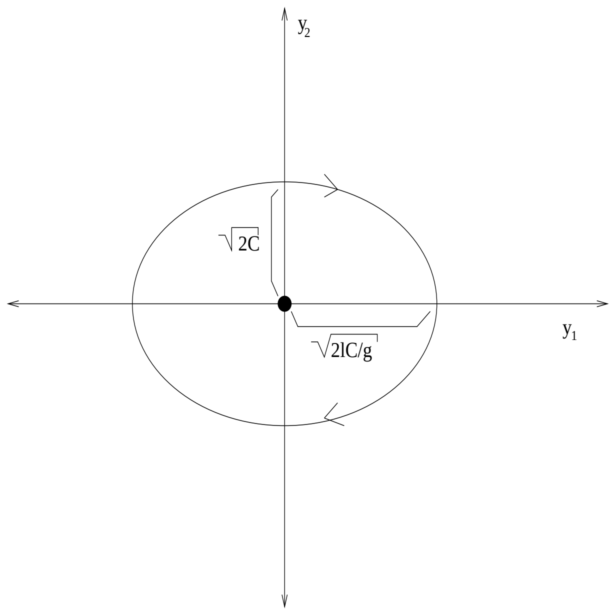

Figure 2.9: Phase plane diagram near the equilibrium point $(0,0)$.

For the point $(\pi, 0)$ we first make the change of variables,

$$y_1 = \pi + z_1$$
$$y_2 = 0 + z_2,$$

and observe that in the variables z_1 and z_2 the equilibrium point is again at $(0,0)$. In these variables, our system becomes,

$$\frac{dz_1}{dt} = z_2$$
$$\frac{dz_2}{dt} = -\frac{g}{l}\sin(\pi + z_1).$$

Recalling the Taylor expansion of $\sin z_1$ at the point π,

$$\sin(\pi + z_1) = \sin\pi + (\cos\pi)z_1 - \frac{\sin\pi}{2}z_1^2 + ...,$$

we arrive at the new linearized equation,

$$\frac{dz_1}{dt} = z_2$$
$$\frac{dz_2}{dt} = \frac{g}{l}z_1.$$

Proceeding exactly as above we again write the phase plane equation,

$$\frac{dz_2}{dz_1} = \frac{\frac{dz_2}{dt}}{\frac{dz_1}{dt}} = \frac{\frac{g}{l}z_1}{z_2},$$

which can be solved by the method of separation of variables for implicit solution,

$$-\frac{z_2^2}{2} + \frac{g}{l}\frac{z_1^2}{2} = C,$$

which corresponds with hyperbolas (see Figure 2.10). Observe that in this case all trajectories move first toward the equilibrium point and then away. We refer to such an equilibrium point as an *unstable saddle*. △

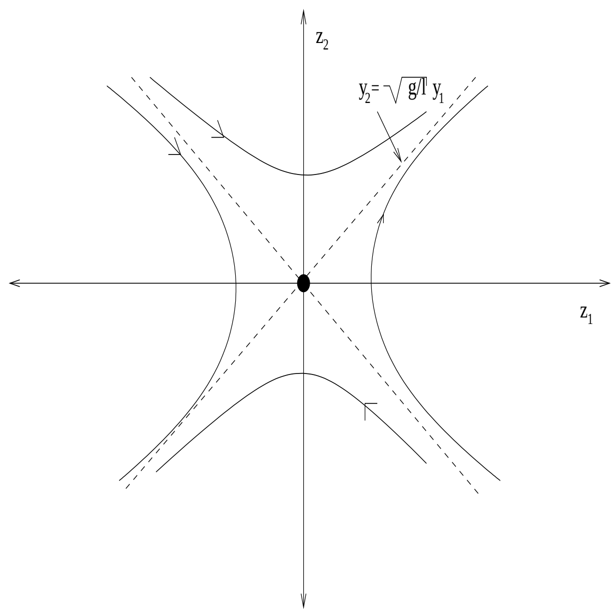

Figure 2.10: Phase plane diagram near the equilibrium point $(\pi, 0)$.

Example 2.3. As a second example of stability analysis by linearization, we will consider the Lotka–Volterra predator–prey equations,

$$\begin{aligned}\frac{dx}{dt} &= ax - bxy \\ \frac{dy}{dt} &= -ry + cxy.\end{aligned} \tag{2.8}$$

First, we find all equilibrium points by solving the system of algebraic equations,

$$\begin{aligned}ax - bxy &= 0 \\ -ry + cxy &= 0.\end{aligned}$$

We find two solutions, $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (\frac{r}{c}, \frac{a}{b})$. The first of these corresponds with an absence of both predator and prey, and the associated linear equation can be obtained simply by observing that the terms $-bxy$ and cxy are higher order. We obtain the linear system

$$\begin{aligned} x' &= ax \\ y' &= -ry, \end{aligned}$$

for which the phase plane equation is

$$\frac{dy}{dx} = -\frac{ry}{ax}.$$

Solving this by separation of variables, we find

$$a\frac{dy}{y} = -r\frac{dx}{x} \Rightarrow a\ln|y| = -r\ln|x| + C \Rightarrow |y| = K|x|^{-r/a},$$

where $K = e^{C/a}$ is a positive constant. If we assume that the initial populations $x(0) = x_0$ and $y(0) = y_0$ are both positive, then the integral curves are described by

$$y(x) = Kx^{-r/a},$$

which have the general form of the curve depicted in Figure 2.11. We conclude that the equilibrium point $(0, 0)$ is unstable.

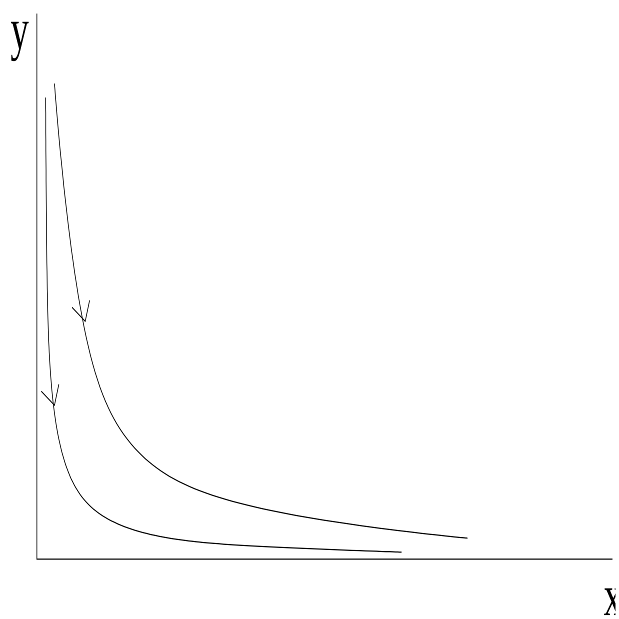

Figure 2.11: Phase plane integral curves for the Lotka-Volterra equation near $(0, 0)$.

The second equilibrium point, $(\frac{r}{c}, \frac{a}{b})$, is more interesting; it is a point at which the predator population and the prey population live together without either one changing. If this second point is unstable then any small fluctuation in either species will destroy the equilibrium and one of the populations will change dramatically. If this second point is stable then small fluctuations in species population will not destroy the equilibrium, and we would expect to observe such equilibria in nature. In this way, *stability typically determines physically viable behavior.*

In order to study the stability of this second point, we first linearize our equations by making the substitutions

$$\begin{aligned} x &= \frac{r}{c} + z_1 \\ y &= \frac{a}{b} + z_2. \end{aligned}$$

Substituting x and y directly into equation (2.8) we find

$$\begin{aligned} \frac{dz_1}{dt} &= a(\frac{r}{c} + z_1) - b(\frac{r}{c} + z_1)(\frac{a}{b} + z_2) = -\frac{br}{c} z_2 - b z_1 z_2 \\ \frac{dz_2}{dt} &= -r(\frac{a}{b} + z_2) + c(\frac{r}{c} + z_1)(\frac{a}{b} + z_2) = \frac{ca}{b} z_1 + c z_1 z_2. \end{aligned}$$

(Observe that in the case of polynomials a Taylor expansion emerges from the algebra, saving us a step.) Dropping the nonlinear terms, we arrive at our linear equations,

$$\begin{aligned} \frac{dz_1}{dt} &= -\frac{br}{c} z_2 \\ \frac{dz_2}{dt} &= \frac{ca}{b} z_1. \end{aligned}$$

Proceeding as in the previous case, we solve the phase plane equation,

$$\frac{dz_2}{dz_1} = \frac{\frac{ca}{b} z_1}{-\frac{br}{c} z_2},$$

for implicit solutions,

$$\frac{ca}{b}\frac{z_1^2}{2} + \frac{br}{c}\frac{z_2^2}{2} = C,$$

which correspond with ellipses and conseqently orbital stability. Just as in the case of the pendulum equation, these orbits correspond with periodic behavior. △

2.3 Linearization and the Jacobian Matrix

In this Section we will develop a more systematic method for obtaining the linear equation associated with an equilibrium point. We begin by recalling what it means for a function of two variables to be differentiable.

Definition 2.4. *We say that $f(x, y)$ is differentiable at the point (x_0, y_0) if the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist, and additionally*

$$\lim_{(h,k)\to(0,0)} \frac{|f(x_0 + h, y_0 + k) - f(x_0, y_0) - f_x(x_0, y_0)h - f_y(x_0, y_0)k|}{\sqrt{h^2 + k^2}} = 0.$$

The derivative of $f(x,y)$ is the gradient vector

$$\nabla f = [f_x, f_y].$$

Equivalently, $f(x,y)$ is differentiable at (x_0,y_0) if there exists some $\epsilon = \epsilon(h,k;x_0,y_0)$ so that

$$f(x_0+h, y_0+k) = f(x_0,y_0) + f_x(x_0,y_0)h + f_y(x_0,y_0)k + \epsilon(h,k;x_0,y_0)\sqrt{h^2+k^2},$$

where $\epsilon \to 0$ as $(h,k) \to (0,0)$.

Consider now a system of two first order autonomous ODE

$$\begin{aligned} y_1' &= f_1(y_1,y_2) \\ y_2' &= f_2(y_1,y_2), \end{aligned}$$

and let $(\hat{y}_1,\hat{y}_2)$ denote an equilibrium point for this system; i.e., $f_1(\hat{y}_1,\hat{y}_2) = f_2(\hat{y}_1,\hat{y}_2) = 0$. If $f_1(y_1,y_2)$ is differentiable at the point $(\hat{y}_1,\hat{y}_2)$ then if, referring to our definition of differentiability, we take $\hat{y}_1 = x_0$, $\hat{y}_2 = y_0$, $y_1 = x_0 + h$, and $y_2 = y_0 + k$, we have

$$f_1(y_1,y_2) = f_1(\hat{y}_1,\hat{y}_2) + \frac{\partial f_1}{\partial y_1}(\hat{y}_1,\hat{y}_2)(y_1-\hat{y}_1) + \frac{\partial f_1}{\partial y_2}(\hat{y}_1,\hat{y}_2)(y_2-\hat{y}_2) + \epsilon\sqrt{(y_1-\hat{y}_1)^2+(y_2-\hat{y}_2)^2}.$$

We observe now that $f(\hat{y}_1,\hat{y}_2) = 0$, and that the term involving ϵ is precisely the type of higher order term that we drop off in the linearization process. We conclude that for (y_1,y_2) near $(\hat{y}_1,\hat{y}_2)$,

$$f_1(y_1,y_2) \approx \frac{\partial f_1}{\partial y_1}(\hat{y}_1,\hat{y}_2)(y_1-\hat{y}_1) + \frac{\partial f_1}{\partial y_2}(\hat{y}_1,\hat{y}_2)(y_2-\hat{y}_2).$$

Proceeding similarly now for f_2, we find that the linear system associated with this equilibrium point is

$$\begin{aligned} y_1' &= \frac{\partial f_1}{\partial y_1}(\hat{y}_1,\hat{y}_2)(y_1-\hat{y}_1) + \frac{\partial f_1}{\partial y_2}(\hat{y}_1,\hat{y}_2)(y_2-\hat{y}_2) \\ y_2' &= \frac{\partial f_2}{\partial y_1}(\hat{y}_1,\hat{y}_2)(y_1-\hat{y}_1) + \frac{\partial f_2}{\partial y_2}(\hat{y}_1,\hat{y}_2)(y_2-\hat{y}_2). \end{aligned}$$

In system form, with perturbation variable

$$\vec{z} = \begin{pmatrix} y_1 - \hat{y}_1 \\ y_2 - \hat{y}_2 \end{pmatrix},$$

this becomes

$$\frac{d\vec{z}}{dt} = \vec{f}'(\hat{y}_1,\hat{y}_2)\vec{z}, \tag{2.9}$$

where $\vec{f}'$ is the *Jacobian* matrix

$$\vec{f}'(\hat{y}_1,\hat{y}_2) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\hat{y}_1,\hat{y}_2) & \frac{\partial f_1}{\partial y_2}(\hat{y}_1,\hat{y}_2) \\ \frac{\partial f_2}{\partial y_1}(\hat{y}_1,\hat{y}_2) & \frac{\partial f_2}{\partial y_2}(\hat{y}_1,\hat{y}_2) \end{pmatrix}.$$

Students may recall from calculus that this matrix is precisely what is meant by the derivative of a two-component vector function that depends on two variables. Hence the prime notation. For our purposes this prime notation will suffice, and it certainly has the advantage of familiarity, but it's generally better to use an operator notation for functions of multiple variables. For example, the standard PDE reference [Evans] uses D for the Jacobian operator, so

$$D\vec{f}(\hat{y}_1, \hat{y}_2) = \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(\hat{y}_1, \hat{y}_2) & \frac{\partial f_1}{\partial y_2}(\hat{y}_1, \hat{y}_2) \\ \frac{\partial f_2}{\partial y_1}(\hat{y}_1, \hat{y}_2) & \frac{\partial f_2}{\partial y_2}(\hat{y}_1, \hat{y}_2) \end{pmatrix}.$$

Example 2.1 revisited. Though the linearization step in Example 2.1 isn't difficult, it will be instructive to see what it looks like with the Jacobian. In the current notation the pendulum system is

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -\frac{g}{l}\sin y_1. \end{aligned}$$

That is, $f_1(y_1, y_2) = y_2$ and $f_2(y_1, y_2) = -\frac{g}{l}\sin y_1$. The Jacobian matrix is

$$\vec{f}'(y_1, y_2) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos y_1 & 0 \end{pmatrix}.$$

We immediately see now that for the equilibrium point $(0, 0)$ we have

$$\vec{f}'(0, 0) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{pmatrix},$$

while for the equilibrium point $(\pi, 0)$ we have

$$\vec{f}'(\pi, 0) = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix}.$$

We see now that in each case (2.9) becomes the same linear system as we derived previously. △

The simplification in the analysis is more marked in the case of Example 2.3 involving the Lotka–Volterra model.

Example 2.3 revisited. In the notation of this section the Lotka–Volterra model has the form

$$\begin{aligned} y_1' &= ay_1 - by_1y_2 \\ y_2' &= -ry_2 + cy_1y_2, \end{aligned}$$

from which we see that $f_1(y_1, y_2) = ay_1 - by_1y_2$ and $f_2(y_1, y_2) = -ry_1 + cy_1y_2$. The Jacobian for this system is

$$\vec{f}'(y_1, y_2) = \begin{pmatrix} a - by_2 & -by_1 \\ cy_2 & -r + cy_1 \end{pmatrix}.$$

For the two equilibrium points $(0,0)$ and $(\frac{r}{c}, \frac{a}{b})$ we have respectively

$$\vec{f}'(0,0) = \begin{pmatrix} a & 0 \\ 0 & -r \end{pmatrix},$$

and

$$\vec{f}'(\frac{r}{c}, \frac{a}{b}) = \begin{pmatrix} 0 & -\frac{br}{c} \\ \frac{ac}{b} & 0 \end{pmatrix}.$$

In each case the system (2.9) becomes precisely the linear system analyzed in Example 2.3. △

More generally, we have the following definition for a function f of n variables. In this case, it is more convenient to state the definition in a vector form, taking the n independent variables as the vector $\vec{x} = (x_1, x_2, \dots, x_n)$.

Definition 2.5. *We say that a function f of n variables $f(\vec{x})$ is differentiable at the point $\vec{x}_0$ if the first partial derivatives of f all exist, and if*

$$\lim_{|\vec{h}| \to 0} \frac{|f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - \nabla f(\vec{x}_0) \cdot \vec{h}|}{|\vec{h}|} = 0.$$

Equivalently, $f(\vec{x})$ is differentiable at $\vec{x}_0$ if there exists some $\epsilon = \epsilon(\vec{h}; \vec{x}_0)$ so that

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot \vec{h} + \epsilon(\vec{h}; \vec{x}_0)|\vec{h}|,$$

where $\epsilon \to 0$ as $|\vec{h}| \to 0$. Here, $|\cdot|$ denotes Euclidean norm, which of course is simply absolute value when applied to a scalar quantity such as $f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - \nabla f(\vec{x}_0) \cdot \vec{h}$.

Proceeding similarly as in the case of a system of two equations we find that if the system of n ODE

$$\frac{d\vec{y}}{dt} = \vec{f}(\vec{y}), \quad \vec{y}, \vec{f} \in \mathbb{R}^n$$

is linearized about the equilibrium point $\vec{y}_e$, with perturbation variable $\vec{z} = \vec{y} - \vec{y}_e$, then the linearized equation is

$$\frac{d\vec{z}}{dt} = f'(\vec{y}_0)\vec{z},$$

where $\vec{f}'$ is the Jacobian of $\vec{f}$. That is,

$$\vec{f}' = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}. \tag{2.10}$$

As mentioned above it is often more convenient to replace the notation $\vec{f}'$ with the operator notation $D\vec{f}$.

Example 2.4. Consider the system of three ODE

$$
\begin{aligned}
y_1' &= ay_1(1 - \frac{y_1}{K}) - by_1y_2 \\
y_2' &= -ry_2 + cy_1y_2 - ey_2y_3 \\
y_3' &= -fy_3 + gy_2y_3,
\end{aligned}
$$

which models a situation in which y_1 denotes a prey population, y_2 denotes the population of a predator that preys on y_1 and y_3 denotes the population of a predator that preys on y_2. Find all equilibrium points $(\hat{y}_1, \hat{y}_2.\hat{y}_3)$ for this system and write down the linearized equation associated with each.

The equilibrium points for this system satisfy the nonlinear system of algebraic equations

$$
\begin{aligned}
\hat{y}_1[a(1 - \frac{\hat{y}_1}{K}) - b\hat{y}_2] &= 0 \\
\hat{y}_2[-r + c\hat{y}_1 - e\hat{y}_3] &= 0 \\
\hat{y}_3[-f + g\hat{y}_2] &= 0.
\end{aligned}
$$

Let's begin with the important case for which the populations are all non-zero, which corresponds with a situation in which the populations can mutually coexist. In this case the third equation gives $\hat{y}_2 = f/g$, and upon substitution of this into the first equation we find

$$
\hat{y}_1 = \frac{K}{a}(a - \frac{bf}{g}).
$$

And finally, from the third equation we get

$$
\hat{y}_3 = \frac{-r + \frac{cK}{a}(a - \frac{bf}{g})}{e}.
$$

For each other equilibrium point at least one of the components will be zero. In the case that $\hat{y}_1 = 0$, we have

$$
\begin{aligned}
\hat{y}_2[-r - e\hat{y}_3] &= 0 \\
\hat{y}_3[-f + g\hat{y}_2] &= 0,
\end{aligned}
$$

from which we obtain $(0,0,0)$, $(0, \frac{f}{g}, -\frac{r}{e})$, the second of which is irrelevant since one of the populations is negative. In the event that $\hat{y}_2 = 0$, we have from the third equation that $\hat{y}_3 = 0$ and

$$
\hat{y}_1[a(1 - \frac{\hat{y}_1}{K})] = 0.
$$

Since all cases for which $\hat{y}_1$ have already been considered, we need only consider $\hat{y}_1 \neq 0$, and here we conclude the new equilibrium point $(K, 0, 0)$. Finally, if $\hat{y}_3 = 0$, we have

$$
\begin{aligned}
\hat{y}_1[a(1 - \frac{\hat{y}_1}{K}) - b\hat{y}_2] &= 0 \\
\hat{y}_2[-r + c\hat{y}_1] &= 0.
\end{aligned}
$$

At this point we have already considered all cases for which either $\hat{y}_1 = 0$ or $\hat{y}_2 = 0$, and so we need only consider the cases in which neither of these populations is 0. We obtain the new equilibrium point $(\frac{r}{c}, \frac{a-\frac{ar}{cK}}{b}, 0)$. Altogether, we have four physical equilibrium points

$$(\frac{K}{a}(a-\frac{bf}{g}), \frac{f}{g}, \frac{-r+\frac{cK}{a}(a-\frac{bf}{g})}{e}), \quad (0,0,0), \quad (K,0,0), \quad (\frac{r}{c}, \frac{a-\frac{ar}{cK}}{b}, 0).$$

The Jacobian for this system is

$$\vec{f}'(\hat{y}_1,\hat{y}_2,\hat{y}_3) = \begin{pmatrix} a - \frac{2a}{K}\hat{y}_1 - b\hat{y}_2 & -b\hat{y}_1 & 0 \\ c\hat{y}_2 & -r + c\hat{y}_1 - e\hat{y}_3 & -e\hat{y}_2 \\ 0 & g\hat{y}_3 & -f + g\hat{y}_2 \end{pmatrix}. \tag{2.11}$$

The linearized equations are then

$$\vec{z}' = f'(\hat{y}_1,\hat{y}_2,\hat{y}_3)z,$$

with the equilibrium point of interest substituted in for $(\hat{y}_1,\hat{y}_2,\hat{y}_3)$. (We will evaluate the Jacobian matrix at these values and determine whether or not these equilbrium points are stable in Subsection 2.6.) △

I'll close this section by reminding students of the role played by the Jacobian in multivariate calculus as the derivative of a vector function of the vector variable.

Definition 2.6. *We say that a vector function $\vec{f}$ of n variables $\vec{f}(\vec{x})$ is differentiable at the point $\vec{x}_0$ if the first partial derivatives of all components of $\vec{f}$ all exist, and if*

$$\lim_{|\vec{h}|\to 0} \frac{|\vec{f}(\vec{x}_0+\vec{h}) - \vec{f}(\vec{x}_0) - \vec{f}'(\vec{x}_0)\vec{h}|}{|\vec{h}|} = 0.$$

Equivalently, $\vec{f}(\vec{x})$ is differentiable at $\vec{x}_0$ if there exists some $\vec{\epsilon} = \vec{\epsilon}(\vec{h};\vec{x}_0)$ so that

$$\vec{f}(\vec{x}_0+\vec{h}) = \vec{f}(\vec{x}_0) + \vec{f}'(\vec{x}_0)\vec{h} + \vec{\epsilon}(\vec{h};\vec{x}_0)|\vec{h}|,$$

where $|\vec{\epsilon}| \to 0$ as $|\vec{h}| \to 0$. Here, $|\cdot|$ denotes Euclidean norm.

In this definition, $\vec{f}'(\vec{x}_0)$ is precisely the Jacobian defined in (2.10). Since this is a matrix the expression $\vec{f}'(\vec{x}_0)\vec{h}$ denotes a matrix multiplying a vector. In this way, the derivative can be regarded as an operator (a matrix is a linear operator) or as a second order tensor.

2.4 Brief Review of Linear Algebra

In this section we will review some ideas from linear algebra that will be important in the development of a more general method for analyzing the stability of equilibrium points. We begin with the determinant of a matrix. First, recall that for a general 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

the inverse is

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}.$$

Clearly, A^{-1} exists if and only if this difference $a_{11}a_{22} - a_{12}a_{21} \neq 0$. This difference is referred to as the determinant of A, and we write

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

More generally, we have the following definition.

Definition 2.7. *For an $n \times n$ matrix A, with entries $\{a_{ij}\}$, $i = 1, 2, ..., n$, $j = 1, 2, ..., n$, the determinant is defined as*

$$\det A = \sum_{permutations} (-1)^{\alpha_p} a_{1p_1} a_{2p_2} \cdots a_{np_n}, \tag{2.12}$$

where the sum is taken over all permutations $(p_1, p_2, \dots, p_n)$ of $(1, 2, \dots, n)$ and α_p denotes the minimum number of exchanges required to put $(p_1, p_2, \dots, p_n)$ in the form $(1, 2, \dots, n)$.

For example, in the case that A is a 2×2 matrix, we can easily list all possible permutations of the indices $(1, 2)$ as $\{(1, 2), (2, 1)\}$. For the first of these $\alpha = 0$, while for the second $\alpha = 1$. In this case, the sum (2.12) becomes

$$(-1)^0 a_{11}a_{22} + (-1)^1 a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}.$$

As a second example, consider the case of a 3×3 matrix for which the possible permuations of the indices $(1, 2, 3)$ are $\{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}$. (In general there will be $n!$ permutations.) In this case the sum (2.12) becomes

$$\begin{aligned} &(-1)^0 a_{11}a_{22}a_{33} + (-1)^1 a_{11}a_{23}a_{32} + (-1)^1 a_{12}a_{21}a_{33} \\ +&(-1)^2 a_{12}a_{23}a_{31} + (-1)^2 a_{13}a_{21}a_{32} + (-1)^1 a_{13}a_{22}a_{31} \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33} + a_{13}a_{22}a_{31}). \end{aligned}$$

Fortunately, there are some properties of determinants that allow us to avoid explicitly using (2.12). In particular, it can be shown that the determinant of an $n \times n$ matrix can always be computed as a sum of determinants of $(n-1) \times (n-1)$ matrices. In this way, the calculation of any general $n \times n$ matrix can be reduced (eventually) to the sum of determinants of 2×2 matrices. Before looking at the general formula, let's consider the case $n = 3$, for which we can write

$$\det A = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{21} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{pmatrix} + a_{31} \det \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{33} \end{pmatrix},$$

or

$$\det A = a_{11}D_{11} - a_{21}D_{21} + a_{31}D_{31},$$

where D_{ij} denotes the determinant of the matrix created by eliminating the i^{th} row and j^{th} column of A. If we define the *cofactor* of element a_{ij} as

$$A_{ij} = (-1)^{i+j} D_{ij},$$

then

$$\det A = \sum_{i=1}^{3} a_{i1} A_{i1}.$$

We call this method *cofactor expansion* along the first column. In fact, the same considerations are true for any $n \times n$ matrix and for expansion along any row or column, and we have the following proposition.

Proposition 2.8. *For any $n \times n$ matrix A with entries $\{a_{ij}\}$, the determinant can be computed either as*

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij},$$

where j denotes any column $j = 1, 2, ..., n$, or as

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij},$$

where i denotes any row $i = 1, 2, ..., n$.

In the next proposition several important properties of determinants are collected.

Proposition 2.9. *If A and B are $n \times n$ matrices and c is a constant, then the following hold:*

$$\begin{aligned}
\det(A^{tr}) &= \det(A) \quad (\textit{here, tr denotes transpose}) \\
\det(AB) &= \det(A)\det(B) \\
\det(A^{-1}) &= \frac{1}{\det A} \quad (\textit{if } A \textit{ is invertible}) \\
\det(cA) &= c^n \det A \\
(A^{-1})_{ij} &= \frac{A_{ji}}{\det A}.
\end{aligned}$$

In the final expression $(A^{-1})_{ij}$ denotes the element of A^{-1} in the i^{th} row and j^{th} column, and A_{ji} is the (transposed) cofactor.

We next review the definitions of linear dependence and linear independence.

Definition 2.10. *The set of n vectors $\vec{v}_1$, $\vec{v}_2$, ..., $\vec{v}_n$ is said to be linearly dependent if there exist values c_1, c_2, ..., c_n, not all 0, so that*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0.$$

If the set of vectors is not linearly dependent then it is called linearly independent.

The following theorem is extremely useful.

Theorem 2.11. *For an $n \times n$ matrix A, the following statements are equivalent.*

1. *A is invertible.*

2. $\det A \neq 0$.

3. *The equation $A\vec{v} = 0$ has only the trivial solution $\vec{v} = \vec{0}$.*

4. *The columns of A form a linearly independent set of n vectors.*

5. *The rows of A form a linearly independent set of n vectors.*

Finally, we recall what is meant by an eigenvalue and by an eigenvector, and we review the method for computing them.

Definition 2.12. *For an $n \times n$ matrix A, the $n \times 1$ column vector $\vec{v}$ is an eigenvector of A if $\vec{v} \neq \vec{0}$ and*

$$A\vec{v} = \lambda\vec{v},$$

for some real or complex value λ. The value λ is called an eigenvalue for A.

In order to see how to compute the eigenvalues of a matrix A, we write the eigenvalue problem in the form

$$(A - \lambda I)\vec{v} = 0,$$

and observe that if the operator $(A - \lambda I)$ is invertible we can compute

$$\vec{v} = (A - \lambda I)^{-1}\vec{0} = \vec{0},$$

in which case $\vec{v}$ is not an eigenvector, and consequently λ is not an eigenvalue. We can conclude from this that λ can only be an eigenvalue of A if the matrix $(A - \lambda I)$ is *not* invertible. In fact, it follows from our theorem on invertibility that λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0.$$

This last equation is often called the *characteristic equation*.

2.5 Solving Linear First-Order Systems

The linearization process discussed in Section 2.3 suggests that at least in certain cases the study of stability can be reduced to the study of linear systems of ODE, which is to say systems of the form

$$\frac{d\vec{y}}{dt} = A\vec{y},$$

where $\vec{y}$ is a column vector with n components and A is an $n \times n$ matrix. In light of this, we need to develop a method for solving such equations.

Example 2.5. Consider again the second linearized equation from Example 2.1 (near the point $(\pi, 0)$),

$$\begin{aligned}\frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= \frac{g}{l}y_1.\end{aligned} \tag{2.13}$$

Recall that we can solve linear autonomous systems such as (2.13) by substitution. That is, writing $y_2 = \frac{dy_1}{dt}$ from the first equation and substituting it into the second, we have the second order equation

$$\frac{d^2y_1}{dt^2} = \frac{g}{l}y_1.$$

Homogeneous constant coefficient equations can be solved through the *ansatz* (guess) $y_1(t) = e^{rt}$, for which we have $r^2 - \frac{g}{l} = 0$, or $r = \pm\sqrt{\frac{g}{l}}$. According to standard ODE theory, we conclude that any solution $y_1(t)$ must have the form

$$y_1(t) = C_1 e^{-\sqrt{\frac{g}{l}}t} + C_2 e^{\sqrt{\frac{g}{l}}t}.$$

In the case of stability, $y_1(t)$ is expected to approach 0 as t gets large (or in the case of orbital stability, at least remain near 0). Since one summand of $y_1(t)$ grows exponentially as t gets large, we conclude that the point under consideration, $(\pi, 0)$, is unstable.

While the method just discussed is fairly convenient for systems of two equations, it isn't efficient for larger systems, and we would like an approach tailored more toward such cases. Consider, then, the general linear system

$$\frac{d\vec{y}}{dt} = A\vec{y}. \tag{2.14}$$

Our next goal will be to develop a method for finding general solutions for such systems. Though we are developing this in the context of stability analysis, this method is of course of interest in its own right. Motivated by the general idea of our approach for solving a second order ODE, we proceed by looking for vector solutions of the form

$$\vec{y} = \vec{v}e^{\lambda t},$$

where $\vec{v}$ denotes a constant vector and the constant multiplying t has been designated as the Greek letter λ in accordance with fairly standard notation. Upon substitution of $\vec{y}$ into (2.14), and multiplication on both sides by $e^{-\lambda t}$, we obtain the linear system of algebraic equations

$$A\vec{v} = \lambda\vec{v},$$

which is an eigenvalue problem (see Subsection 2.4). In the case of (2.13), we have

$$A = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix},$$

for which

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ \frac{g}{l} & -\lambda \end{pmatrix} = \lambda^2 - \frac{g}{l} = 0 \Rightarrow \lambda = \pm\sqrt{\frac{g}{l}}.$$

We can determine the eigenvectors associated with these eigenvalues by solving

$$A\vec{v} - \lambda\vec{v} = 0 \Rightarrow \begin{pmatrix} \mp\sqrt{\frac{g}{l}} & 1 \\ \frac{g}{l} & \mp\sqrt{\frac{g}{l}} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2 = \pm\sqrt{\frac{g}{l}}v_1.$$

Observe, in particular, that though we have two equations, we only get one relation for each eigenvalue. This means that one component of $\vec{v}$ can be chosen (almost) arbitrarily, which corresponds with the observation that if you multiply an eigenvector by a constant, you will get another (linearly dependent) eigenvector. In this case, let's choose $v_1 = 1$ for each eigenvector (recall that we should have two), giving

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -\sqrt{\frac{g}{l}} \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ \sqrt{\frac{g}{l}} \end{pmatrix}.$$

Here, it is important to observe the distinction between the vector component v_1 and the first eigenvector $\vec{v}_1$. Finally, we can write down the general solution to the ODE system (2.13)

$$\vec{y}(t) = c_1 \begin{pmatrix} 1 \\ -\sqrt{\frac{g}{l}} \end{pmatrix} e^{-\sqrt{\frac{g}{l}}t} + c_2 \begin{pmatrix} 1 \\ \sqrt{\frac{g}{l}} \end{pmatrix} e^{\sqrt{\frac{g}{l}}t}.$$

At this point it is important to understand whether or not every possible solution to (2.13) can be written in this form. That is, given an arbitrary initial vector

$$\vec{y} = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix},$$

can we be sure that there exist values c_1 and c_2 so that

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -\sqrt{\frac{g}{l}} \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ \sqrt{\frac{g}{l}} \end{pmatrix}.$$

Noting that we can rearrange this last equation into the matrix form

$$\begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{g}{l}} & \sqrt{\frac{g}{l}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix},$$

we see that c_1 and c_2 exist if and only if the matrix

$$\begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{g}{l}} & \sqrt{\frac{g}{l}} \end{pmatrix}$$

is invertible. According to our theorem on matrix invertibility, we can check this by computing the determinant of this matrix, which is $2\sqrt{\frac{g}{l}} \neq 0$. And so our general solution will work for any initial condition. In this case, we find

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \sqrt{\frac{l}{4g}} \begin{pmatrix} \sqrt{\frac{g}{l}} & -1 \\ \sqrt{\frac{g}{l}} & 1 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \sqrt{\frac{l}{4g}} \begin{pmatrix} \sqrt{\frac{g}{l}} y_1(0) - y_2(0) \\ \sqrt{\frac{g}{l}} y_1(0) + y(0) \end{pmatrix}.$$

The solution to the initial value problem is consequently

$$\vec{y}(t) = \frac{1}{2}\Big(y_1(0) - \sqrt{\frac{l}{g}} y_2(0)\Big) \begin{pmatrix} 1 \\ -\sqrt{\frac{g}{l}} \end{pmatrix} e^{-\sqrt{\frac{g}{l}}t} + \frac{1}{2}\Big(y_1(0) + \sqrt{\frac{l}{g}} y_2(0)\Big) \begin{pmatrix} 1 \\ \sqrt{\frac{g}{l}} \end{pmatrix} e^{\sqrt{\frac{g}{l}}t}.$$

△

More generally, for the linear constant-coefficient system of n ODE

$$\frac{d\vec{y}}{dt} = A\vec{y},$$

suppose that A has n linearly independent eigenvectors $\{\vec{v}_k\}_{k=1}^n$, with associated eigenvalues $\{\lambda_k\}_{k=1}^n$. Our general solution in this case is

$$\vec{y}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t} = \sum_{k=1}^{n} c_k\vec{v}_k e^{\lambda_k t}.$$

As in the case of Example 2.5, if we are given general initial conditions $\vec{y}(0)$, we can write down a matrix equation for the c_k in which the matrix is composed by taking each eigenvector as a column. According to our theorem on matrix invertibility, the linear independence of the eigenvectors insures that the matrix will be invertible, and consequently that we will be able to find values for the c_k. Collecting these observations, we have the following theorem.

Theorem 2.13. *Suppose A is a constant $n \times n$ matrix with n linearly independent eigenvectors $\{\vec{v}_k\}_{k=1}^n$ and associated eigenvalues $\{\lambda_k\}_{k=1}^n$. Then the initial value problem*

$$\frac{d\vec{y}}{dt} = A\vec{y}; \quad \vec{y}(0) = \vec{y}_0$$

has a unique solution

$$\vec{y}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} + \cdots + c_n\vec{v}_n e^{\lambda_n t} = \sum_{k=1}^{n} c_k\vec{v}_k e^{\lambda_k t},$$

where the $\{c_k\}_{k=1}^n$ are determined by the system of equations

$$\vec{y}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

Example 2.6. Solve the initial value problem

$$\begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= y_1 + 2y_2 + y_3 \\ y_3' &= 3y_2 - y_3, \end{aligned}$$

subject to $y_1(0) = y_2(0) = 0$ and $y_3(0) = 1$.

We begin by writing this system in matrix form $\vec{y}' = A\vec{y}$, where

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}.$$

The eigenvalues of A are roots of the characteristic equation

$$\det(A - \lambda I) = \det \begin{pmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{pmatrix} = 0.$$

If we compute this determinant by the method of cofactor expansion down the first column, we find

$$(-1-\lambda)((2-\lambda)(-1-\lambda)-3)-1(-1-\lambda)=0 \Rightarrow \lambda^3-7\lambda+6=0,$$

where the roots of this cubic polynomial are $\lambda=-2,-1,3$, which are consequently the eigenvalues of A. For the first of these, the eigenvector equation is

$$\begin{pmatrix} -1+2 & 1 & 0 \\ 1 & 2+2 & 1 \\ 0 & 3 & -1+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

or

$$\begin{aligned} v_1+v_2&=0 \\ v_1+4v_2+v_3&=0 \\ 3v_2+v_3&=0. \end{aligned}$$

Observe particularly that if we subtract the first equation from the second we obtain the third, so the third equation can be regarded as superfluous, giving us the degree of freedom we expect. If we now choose $v_1=1$, we can conclude $v_2=-1$ and $v_3=3$, so that

$$\vec{v}_1=\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}.$$

Proceeding similarly, we find that the eigenvectors associated with $\lambda_2=-1$ and $\lambda_3=3$ are respectively

$$\vec{v}_2=\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{v}_3=\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}.$$

In this way, the general solution for this system is

$$\vec{y}(t)=c_1\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}e^{-2t}+c_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t}+c_3\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}e^{2t}.$$

In order to find values for c_1, c_2, and c_3, we use the initial conditions, which give the system $P\vec{c}=\vec{y}_0$, where

$$P=\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 4 \\ 3 & -1 & 3 \end{pmatrix}.$$

Solving either by row reduction or matrix inversion, we conclude $c_1=1/5$, $c_2=-1/4$, and $c_3=1/20$, which gives

$$\vec{y}(t)=\frac{1}{5}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}e^{-2t}-\frac{1}{4}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}e^{-t}+\frac{1}{20}\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}e^{2t}.$$

△

2.6 Stability and Eigenvalues

Recall from Subsection 2.3 that if $\vec{y}_e$ is an equilibrium point for the autonomous ODE system $\vec{y}' = \vec{f}(\vec{y})$, then the associated linear system is

$$\frac{d\vec{z}}{dt} = A\vec{z}. \tag{2.15}$$

where A is the Jacobian matrix $\vec{f}'(\vec{y}_e)$. We see immediately that $\vec{z}_e = \vec{0}$ is an equilibrium point for this system, and in fact by our theorem on matrix invertibility that if A is invertible this will be the only equilibrium point for this system. In this section, we combine elements of Subsections 2.3, 2.4, and 2.5 to arrive at a general theorem on the stability of $\vec{z}_e = \vec{0}$ as a solution to $\vec{z}' = A\vec{z}$. We will begin in the case of two equations, for which it is possible to relate the spectral result to our phase plane analysis of Subsection 2.1.

Example 2.7. For the ODE system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \frac{g}{l} y_1, \end{aligned}$$

use eigenvalues to determine whether or not the equilibrium point $\vec{y}_e = \vec{0}$ is stable.

For this system, we have already shown in Subsection 2.5 that that for general initial values $y_1(0)$ and $y_2(0)$ the solution is

$$\vec{y}(t) = \frac{1}{2}\Big(y_1(0) - \sqrt{\frac{l}{g}} y_2(0)\Big) \begin{pmatrix} 1 \\ -\sqrt{\frac{g}{l}} \end{pmatrix} e^{-\sqrt{\frac{g}{l}}t} + \frac{1}{2}\Big(y_1(0) + \sqrt{\frac{l}{g}} y_2(0)\Big) \begin{pmatrix} 1 \\ \sqrt{\frac{g}{l}} \end{pmatrix} e^{\sqrt{\frac{g}{l}}t}.$$

We recall now that according to our definition of stability, $\vec{y}_e = \vec{0}$ will be stable if by choosing $y_1(0)$ and $y_2(0)$ so that $|\vec{y}(0)| > 0$ is sufficiently small, we can insure that $|\vec{y}(t)|$ remains small for all t (stability) or decays to 0 as $t \to \infty$ (asymptotic stability). In this case, we can see from our exact solution that for general values of $y_1(0)$ and $y_2(0)$ the second summand in $\vec{y}(t)$ will grow at exponential rate in t, and so no matter how small $|\vec{y}(0)| > 0$ is $|\vec{y}(t)|$ will eventually grow large. Observe particularly, however, that there is one choice of initial conditions for which this growth will not occur, namely $y_2(0) = -\sqrt{\frac{g}{l}} y_1(0)$, which specifies a line through the origin in the phase plane (see Figure 2.12). The critical thing to observe about this line is that it corresponds precisely with the eigenvector associated with the negative eigenvalue $-\sqrt{\frac{g}{l}}$. That is, the eigenvector

$$\begin{pmatrix} 1 \\ -\sqrt{\frac{g}{l}} \end{pmatrix}$$

points in the same direction as does the line described by $y_2(0) = -\sqrt{\frac{g}{l}} y_1(0)$. (Note that the small arrow on this line in Figure 2.12 points in the direction of the vector, while the large arrow points in the direction $y_1(t)$ and $y_2(t)$ move as t increases.) Likewise, if $y_2(0) = \sqrt{\frac{g}{l}} y_1(0)$ then the first summand in $\vec{y}(t)$ is 0, and the solution *only* has a growing term. That is, the eigenvector associated with the positive eigenvalue $\sqrt{\frac{g}{l}}$ points in a direction of pure

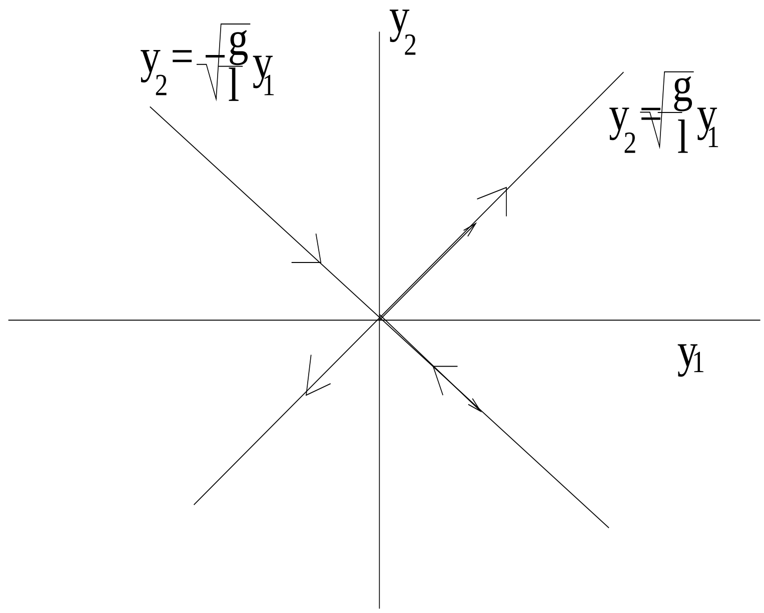

Figure 2.12: Eigenvectors for the pendulum equation linearized about $(\pi, 0)$.

growth. This gives the partial phase plane diagram depicted in Figure 2.12, from which we conclude that the equilibrium point $(0,0)$ is an unstable saddle for this system. △

We have seen in Subsection 2.5 that if $\{\vec{v}_k\}_{k=1}^n$ denotes a linearly independent set of eigenvectors for A, the general solution to $\vec{y}' = A\vec{y}$ has the form

$$\vec{y}(t) = \sum_{k=1}^{n} c_k \vec{v}_k e^{\lambda_k t}.$$

In this way, the general solution for the linear perturbation variable z of (2.15) is

$$\vec{z}(t) = \sum_{k=1}^{n} c_k \vec{v}_k e^{\lambda_k t},$$

where $\{v_k\}_{k=1}^n$ are eigenvectors of the Jacobian matrix (assumed linearly independent) and $\{\lambda_k\}_{k=1}^n$ denote the corresponding eigenvalues. In this way it's clear that $|\vec{z}| \to 0$ as $t \to \infty$ (i.e., asymptotic stability) if and only if the eigenvalues all have negative real part. Also, it's clear that if any of the eigenvalues have positive real part then for general initial conditions $|\vec{z}(t)| \to \infty$ as $t \to \infty$. Accordingly, we have the following theorem.

Theorem 2.14. *(Linear ODE Stability) For the linear first order constant coefficient system of ODE*

$$\frac{d\vec{y}}{dt} = A\vec{y}, \quad \vec{y} \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n},$$

the zero vector $\vec{y}_e \equiv 0$ is stable or unstable as follows:

1. *If all eigenvalues of A have nonpositive real parts, and all those with zero real parts are simple, then $\vec{y}_e = 0$ is stable.*

2. If and only if all eigenvalues of A have negative real parts, then $\vec{y}_e = 0$ is asymptotically stable.

3. If one or more eigenvalues of A have a postive real part, then $\vec{y}_e = 0$ is unstable.

Definition 2.15. *For the general autonomous ODE,*

$$\frac{d\vec{y}}{dt} = \vec{f}(\vec{y}), \quad \vec{y}, \vec{f} \in \mathbb{R}^n, \tag{2.16}$$

with equilibrium point $\vec{y}_e$, if the linearization of (2.16) about $\vec{y}_e$ yields a linear equation for which the zero vector is stable, then we refer to $\vec{y}_e$ as linearly stable.

In the case of a system of two equations, we can say more. Proceeding as in Example 2.7, we can associate partial phase diagrams with eigenvalues in the following way. Let A be a 2×2 matrix with eigenvalues λ_1 and λ_2. Then the equilibrium point $(0,0)$ can be categorized as follows[1]:

1. If $\lambda_1 < \lambda_2 < 0$ then $(0,0)$ is asymptotically stable and we refer to it as a *sink*.

2. If $0 < \lambda_1 < \lambda_2$ then $(0,0)$ is unstable and we refer to it as a *source*.

3. If $\lambda_1 < 0 < \lambda_2$ then $(0,0)$ is unstable and we refer to it as a *saddle*.

4. If $\lambda_1 = -ib$, $\lambda_2 = +ib$ then $(0,0)$ is stable and we refer to it as a *center* or *orbit*.

5. If $\lambda_1 = -a - ib$, $\lambda_2 = -a + ib$ (for $a > 0$) then $(0,0)$ is asymptotically stable and we refer to it as a *stable spiral*.

6. If $\lambda_1 = a - ib$, $\lambda_2 = a + ib$ (for $a > 0$) then $(0,0)$ is unstable and we refer to it as an *unstable spiral*.

7. If $\lambda_1 = \lambda_2 < 0$, *and* the single eigenvalue has two linearly independent eigenvectors then $(0,0)$ is asymptotically stable and we refer to it as a *stable proper node*.

8. If $\lambda_1 = \lambda_2 < 0$, *and* the single eigenvalue does not have two linearly independent eigenvectors then $(0,0)$ is asymptotically stable and we refer to it as a *stable improper node*.

9. If $\lambda_1 = \lambda_2 > 0$, *and* the single eigenvalue has two linearly independent eigenvectors then $(0,0)$ is unstable and we refer to it as an *unstable proper node*.

10. If $\lambda_1 = \lambda_2 > 0$, *and* the single eigenvalue does not have two linearly independent eigenvectors then $(0,0)$ is unstable and we refer to it as an *unstable improper node*.

The six most common cases are depicted in Figure 2.13.

Example 2.8. Find and characterize all equilibrium points for the competition model

$$\begin{aligned} \frac{dy_1}{dt} &= r_1 y_1 (1 - \frac{y_1 + s_1 y_2}{K_1}) \\ \frac{dy_2}{dt} &= r_2 y_2 (1 - \frac{y_2 + s_2 y_1}{K_2}), \end{aligned}$$

assuming all parameter values are positive.

[1] This terminology is not universal; we follow [BD]

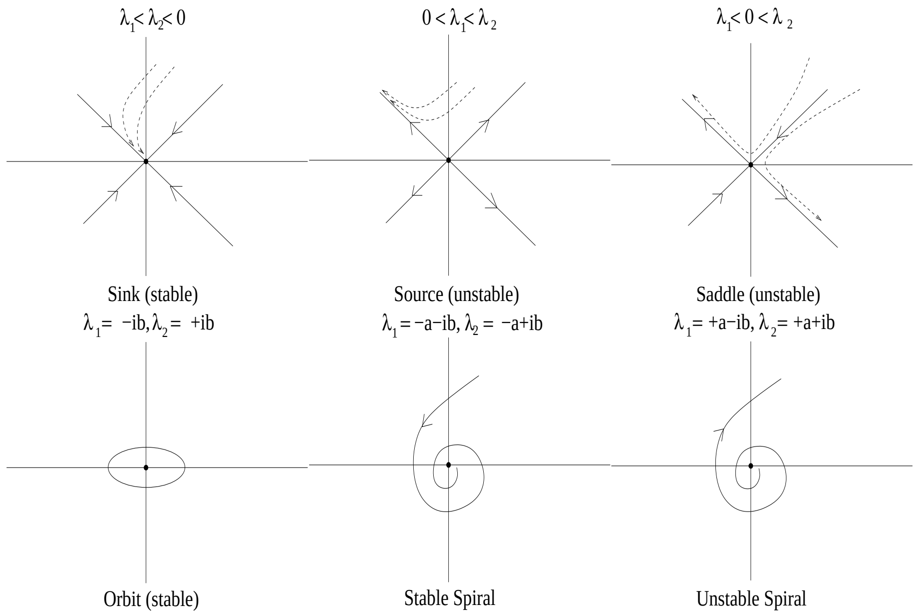

Figure 2.13: Eight common types of equilibrium points for systems of 2 equations.

The equilibrium points $(\hat{y}_1, \hat{y}_2)$ satisfy the algebraic system of equations

$$\begin{aligned} r_1\hat{y}_1(1 - \frac{\hat{y}_1 + s_1\hat{y}_2}{K_1}) &= 0 \\ r_2\hat{y}_2(1 - \frac{\hat{y}_2 + s_2\hat{y}_1}{K_2}) &= 0. \end{aligned}$$

For the first of these equations we must have either $\hat{y}_1 = 0$ or $\hat{y}_1 = K_1 - s_1\hat{y}_2$, while for the second we must have either $\hat{y}_2 = 0$ or $\hat{y}_2 = K_2 - s_2\hat{y}_1$. In this way, we can proceed in four cases, substituting each solution of the first equation into the solution equation for the second. The first pair is $\hat{y}_1 = 0$ and $\hat{y}_2 = 0$, which clearly gives the equilibrium point $(0, 0)$. The second pair is $\hat{y}_1 = 0$ and $\hat{y}_2 = K_2$, giving $(0, K_2)$. The third pair is $\hat{y}_1 = K_1 - s_1\hat{y}_2$ with $\hat{y}_2 = 0$, giving $(K_1, 0)$ and the final pair is $\hat{y}_1 = K_1 - s_1\hat{y}_2$ with $\hat{y}_2 = K_2 - s_2\hat{y}_1$, which can be solved to give $(\frac{K_1 - s_1K_2}{1 - s_1s_2}, \frac{K_2 - s_2K_1}{1 - s_2s_2})$. (Notice that for certain parameter values these populations can be negative, which simply means that for certain parameter values there are no equilibrium points for which both populations are present.)

The Jacobian matrix for this system is

$$\vec{f}'(\hat{y}_1, \hat{y}_2) = \begin{pmatrix} r_1 - 2\frac{r_1}{K_1}\hat{y}_1 - \frac{r_1s_1}{K_1}\hat{y}_2 & -\frac{r_1s_1}{K_1}\hat{y}_1 \\ -\frac{r_2s_2}{K_2}\hat{y}_2 & r_2 - 2\frac{r_2}{K_2}\hat{y}_2 - \frac{r_2s_2}{K_2}\hat{y}_1 \end{pmatrix}.$$

In order to determine whether or not each of our equilibrium points is (linearly) stable, we need to compute the eigenvalues of the Jacobian matrix evaluated at that equilibrium point. For the point $(0, 0)$, we clearly have

$$\vec{f}'(0, 0) = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix},$$

with eigenvalues r_1 and r_2. Since r_1 and r_2 are both assumed positive, we can conclude that this equilibrium point is unstable. More precisely, it is a source.

For the equilibrium point $(K_1, 0)$, the Jacobian matrix is

$$\vec{f}'(K_1, 0) = \begin{pmatrix} -r_1 & -r_1 s_1 \\ 0 & r_2 - \frac{r_2 s_2}{K_2} K_1 \end{pmatrix},$$

with eigenvalues $-r_1$ and $r_2(1 - s_2 \frac{K_1}{K_2})$. In this case, we can conclude instability if $s_2 K_1 < K_2$ and stability if $s_2 K_1 \geq K_2$. This is easily interpreted as follows: suppose the populations are $(K_1, 0)$ and that a small population of species 2 is introduced so that we have populations (K_1, ϵ). Then

$$y_2' = r_2 \epsilon (1 - s_2 \frac{K_1}{K_2}) - \frac{r_2}{K_2} \epsilon^2,$$

where the ϵ^2 term is small enough to be ignored (this is precisely what we ignore in the linearization process). We see that y_2 grows precisely when $1 - s_2 \frac{K_1}{K_2} > 0$, which is our instability condition.

For the equilibrium point $(0, K_2)$, the Jacobian matrix is

$$\vec{f}'(0, K_2) = \begin{pmatrix} r_1 - \frac{r_1 s_1}{K_1} K_2 & 0 \\ -r_2 s_2 & -r_2 \end{pmatrix},$$

with eigenvalues $r_1(1 - s_1 \frac{K_2}{K_1})$ and $-r_2$. Proceeding similarly as with the point $(K_1, 0)$, we conclude that this equilibrium point is linearly stable for $s_1 K_2 \geq K_1$ and linearly unstable for $s_1 K_2 < K_1$.

For the equilibrium point $(\frac{K_1 - s_1 K_2}{1 - s_1 s_2}, \frac{K_2 - s_2 K_1}{1 - s_2 s_2})$, we first observe that the linear system of equations that we solved to find this point can be written in the form

$$\begin{aligned} r_1 - \frac{r_1}{K_1} \hat{y}_1 - \frac{r_1 s_1}{K_1} \hat{y}_2 &= 0 \\ r_2 - \frac{s_2 r_2}{K_2} \hat{y}_1 - \frac{r_2}{K_2} \hat{y}_2 &= 0, \end{aligned}$$

and that these equations allow us to write the Jacobian matrix for this equilibrium point in the simplified form

$$\vec{f}'(\hat{y}_1, \hat{y}_2) = \begin{pmatrix} -\frac{r_1}{K_1} \hat{y}_1 & -\frac{r_1 s_1}{K_1} \hat{y}_1 \\ -\frac{r_2 s_2}{K_2} \hat{y}_2 & -\frac{r_2}{K_2} \hat{y}_2 \end{pmatrix}.$$

The eigenvalues of this matrix are roots of the characteristic equation

$$\det \begin{pmatrix} -\frac{r_1}{K_1} \hat{y}_1 - \lambda & -\frac{r_1 s_1}{K_1} \hat{y}_1 \\ -\frac{r_2 s_2}{K_2} \hat{y}_2 & -\frac{r_2}{K_2} \hat{y}_2 - \lambda \end{pmatrix} = (-\frac{r_1}{K_1} \hat{y}_1 - \lambda)(-\frac{r_2}{K_2} \hat{y}_2 - \lambda) - \frac{r_1 s_1}{K_1} \hat{y}_1 \frac{r_2 s_2}{K_2} \hat{y}_2 = 0,$$

or

$$\lambda^2 + (\frac{r_1 \hat{y}_1}{K_1} + \frac{r_2 \hat{y}_2}{K_2}) \lambda - \frac{r_1 r_2}{K_1 K_2} \hat{y}_1 \hat{y}_2 (1 - s_1 s_2) = 0.$$

According to the quadratic formula, the solutions are

$$\lambda = \frac{-(\frac{r_1 \hat{y}_1}{K_1} + \frac{r_2 \hat{y}_2}{K_2}) \pm \sqrt{(\frac{r_1 \hat{y}_1}{K_1} + \frac{r_2 \hat{y}_2}{K_2})^2 + 4 \frac{r_1 r_2}{K_1 K_2} \hat{y}_1 \hat{y}_2 (1 - s_1 s_2)}}{2}.$$

The only way these eigenvalues can be positive is if the value generated by the radical is larger than $(\frac{r_1\hat{y}_1}{K_1}+\frac{r_2\hat{y}_2}{K_2})$, and with all parameter values positive this only occurs if $1-s_1s_2>0$. Hence, we have linear stability for $1-s_1s_2\leq 0$ and linear instability for $1-s_1s_2>0$. $\triangle$

Example 2.4. cont. For the nonlinear system of equations considered in Example 2.4, eigenvalues of the Jacobian matrix are cumbersome to evaluate in general, and for the two more complicated equilibrium points we will work with specific parameter values. For the points $(0,0,0)$ and $(K,0,0)$, however, we can fairly easily obtain general expressions for the eigenvalues.

In this case, we have already found a general form of the Jacobian matrix in (2.11),

$$\vec{f}'(\hat{y}_1,\hat{y}_2,\hat{y}_3)=\begin{pmatrix} a-\frac{2a}{K}\hat{y}_1-b\hat{y}_2 & -b\hat{y}_1 & 0 \\ c\hat{y}_2 & -r+c\hat{y}_1-e\hat{y}_3 & -e\hat{y}_2 \\ 0 & g\hat{y}_3 & -f+g\hat{y}_2 \end{pmatrix}.$$

For the point $(0,0,0)$, this becomes

$$\vec{f}'(0,0,0)=\begin{pmatrix} a & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & -f \end{pmatrix},$$

and we immediately see that the eigenvalues are a, $-r$, and $-f$. Since $a>0$, the point is unstable. For the point $(K,0,0)$ the Jacobian matrix is

$$\vec{f}'(\hat{y}_1,\hat{y}_2,\hat{y}_3)=\begin{pmatrix} -a & -bK & 0 \\ 0 & -r+cK & 0 \\ 0 & 0 & -f \end{pmatrix},$$

and again we can read off the eigenvalues as $-a$, $-r+cK$, and $-f$. In this case, the equilibrium point is stable so long as $r\geq cK$, and asymptotically stable if $r>cK$. If $r<cK$ the point is unstable. We can interpret this as follows: in the absence of population y_3, we have $y_2'=y_2(cy_1-r)$. Since $y_1=K$ in equilibrium, the population y_2 will grow if $cK-r>0$. (We can focus on the case $y_3=0$ because equilibrium points are unstable if they are unstable to *any* perturbation.)

The expressions become uninstructive if we consider the remaining two equilibrium points in all generality, so let's focus on a choice of parameter values $a=2$, $K=1$, $b=1$, $r=\frac{1}{2}$, $c=4$, $e=2$, $f=1$, and $g=2$. In this case, we have

$$\left(\frac{K}{a}(a-\frac{bf}{g}),\frac{f}{g},\frac{-r+\frac{cK}{a}(a-\frac{bf}{g})}{e}\right)=(\frac{3}{4},\frac{1}{2},\frac{5}{4}) \quad \text{and} \quad (\frac{r}{c},\frac{a-\frac{ar}{cK}}{b},0)=(\frac{1}{8},\frac{7}{4},0).$$

For the second of these, the Jacobian matrix is

$$\vec{f}'(\frac{1}{8},\frac{7}{4},0)=\begin{pmatrix} -\frac{1}{4} & -\frac{1}{8} & 0 \\ 7 & 0 & -\frac{7}{2} \\ 0 & 0 & \frac{5}{2} \end{pmatrix}.$$

The eigenvalues are solutions of the characteristic equation

$$\det\begin{pmatrix} -\frac{1}{4}-\lambda & -\frac{1}{8} & 0 \\ 7 & -\lambda & -\frac{7}{2} \\ 0 & 0 & \frac{5}{2}-\lambda \end{pmatrix}=(\frac{5}{2}-\lambda)\det\begin{pmatrix} -\frac{1}{4}-\lambda & -\frac{1}{8} \\ 7 & -\lambda \end{pmatrix}=0,$$

from which it is clear that $\lambda = \frac{5}{2}$ is one of the eigenvalues so that the equilibrium point is unstable. Finally, for the point $(\frac{3}{4}, \frac{1}{2}, \frac{5}{4})$, for which all populations are non-zero, we have

$$\vec{f}'(\frac{3}{4}, \frac{1}{2}, \frac{5}{4}) = \begin{pmatrix} -\frac{3}{2} & -\frac{3}{4} & 0 \\ 2 & 0 & -1 \\ 0 & \frac{5}{2} & 0 \end{pmatrix}.$$

The eigenvalues are solutions of the characteristic equation

$$\det \begin{pmatrix} -\frac{3}{2} - \lambda & -\frac{3}{4} & 0 \\ 2 & -\lambda & -1 \\ 0 & \frac{5}{2} & -\lambda \end{pmatrix} = (-\frac{3}{2} - \lambda)[\lambda^2 + \frac{5}{2}] - 2[\frac{3}{4}\lambda] = 0,$$

or

$$\lambda^3 + \frac{3}{2}\lambda^2 + 4\lambda + \frac{15}{4} = 0.$$

We can solve this equation in MATLAB as follows.

```
>>eval(solve('x^3+1.5*x^2+4*x+15/4'))
ans =
-0.2195 + 1.8671i
-1.0610
-0.2195 - 1.8671i
```

We see that the real part of each equilibrium point is negative, and so we conclude that this equilibrium point is linearly stable.

As a final remark, I'll observe that if we are going to use MATLAB anyway, we way as well use the *eig* command. That is:

```
>>J=[-3/2 -3/4 0;2 0 -1;0 5/2 0]
J =
-1.5000 -0.7500 0
2.0000 0 -1.0000
0 2.5000 0
>>eig(J)
ans =
-1.0610
-0.2195 + 1.8671i
-0.2195 - 1.8671i
```

$\triangle$

2.7 Linear Stability with MATLAB

In this section we repeat the linear stability analysis for the Lotka–Volterra competition model in Example 2.8, this time using MATLAB. We use the M-file *stability2.m*:

Example 2.8 cont. Given the parameter values $r_1 = r_2 = K_1 = K_2 = 1$, $s_1 = \frac{1}{2}$, and $s_2 = \frac{1}{4}$, find all equilibrium points for the Lotka–Volterra competition model and determine whether or not each is stable.

We use the MATLAB M-file stability2.m.

```
%STABILITY2: MATLAB script M-file that locates
%equilibrium points for the Lotka-Volterra competition
%model, computes the Jacobian matrix for each
%equilibrium point, and finds the eigenvalues of
%each Jacobian matrix. Parameter values are chosen
%as r1=r2=K1=K2=1, s1=.5, s2=.25.
syms y1 y2;
y = [y1, y2];
f=[y1*(1-(y1+y2/2)), y2*(1-(y1/4+y2))];
[ye1 ye2] = solve(f(1),f(2))
J = jacobian(f,y)
for k=1:length(ye1)
y1 = eval(ye1(k))
y2 = eval(ye2(k))
A = subs(J)
lambda = eig(A)
end
```

We find:

```
>>stability2
ye1 =
0
1
0
4/7
ye2 =
0
0
1
6/7
J =
[ 1-2*y1-1/2*y2, -1/2*y1]
[ -1/4*y2, 1-1/4*y1-2*y2]
y1 =
0
```

y2 =
0
A =
1 0
0 1
lambda =
1
1
y1 =
1
y2 =
0
A =
-1.0000 -0.5000
0 0.7500
lambda =
-1.0000
0.7500
y1 =
0
y2 =
1
A =
0.5000 0
-0.2500 -1.0000
lambda =
-1.0000
0.5000
y1 =
0.5714
y2 =
0.8571
A =
-0.5714 -0.2857
-0.2143 -0.8571
lambda =
-0.4286
-1.0000

In this case the equilibrium points are respectively $(0,0)$, $(1,0)$, $(0,1)$, and $(\frac{4}{7}, \frac{6}{7})$, and we can immediately conclude that the first three are unstable while the third is stable. $\triangle$

2.8 Application to Maximum Sustainable Yield

An issue closely related to stability is *maximum sustainable yield*. The maximum sustainable yield for a population of, say, fish, is the maximum number that can be harvested without killing off the population.

Example 2.9. Suppose that in the absence of fishermen the population of fish, $p(t)$, in a certain lake follows a logistic model, and that fishing yield is added as a percent of population. Determine the maximum sustainable yield for this population of fish and describe what will happen to the fish population if the maximum sustainable yield is harvested.

Keeping in mind that as long as we remain at non-zero stable equilibrium points the fish population will not die out, we begin by determining the equilibrium points for our model. Subtracting a percent harvest from the logistic model, we have

$$\frac{dp}{dt} = rp(1 - \frac{p}{K}) - hp,$$

where r and K are as described in Example 2.3 and h is the population harvest rate. In order to find the equilibrium points for this equation, we simply solve

$$rp_e(1 - \frac{p_e}{K}) - hp_e = 0 \Rightarrow p_e = 0, K(1 - \frac{h}{r}).$$

Since population can be regarded as positive, we conclude that a necessary condition for any sustainable yield is $h < r$, which simply asserts that we much catch fewer fish than are born.

In order to apply the eigenvalue approach to the stability of these equilibrium points, we recall that the eigenvalue of a 1×1 matrix (a.k.a. a scalar constant) is just the constant itself. For the equilibrium points $p_e = 0$, we obtain our linearized equation simply by dropping all high order terms, and we find

$$\frac{dp}{dt} = (r - h)p,$$

for which the matix A is the scalar $A = r - h$. But we have already argued that $r > h$, so $A > 0$, and we can conclude instability. For the second equilibrium point, we introduce the perturbation variable $x(t)$ through

$$p(t) = K(1 - \frac{h}{r}) + x(t),$$

for which we find

$$\frac{dx}{dt} = r(x + K(1 - \frac{h}{r}))(1 - \frac{x + K(1 - \frac{h}{r})}{K}) - h(x + K(1 - \frac{h}{r})).$$

Dropping high order terms, we have

$$\frac{dx}{dt} = -(r - h)x,$$

for which $A = -(r - h) < 0$, and we conclude stability. (In the case of single equations, stability is more readily observed directly, through consideration of the sign of $\frac{dp}{dt}$ for p above and below the equilibrium point (see the discussion of the logistic equation in Section 2.3), but it's instructive to see how the general approach works.) We conclude that so long as $h < r$, $p_e = K(1 - \frac{h}{r})$ is an equilibrium point.

Finally, we choose our harvest rate h to maxiumize the yield, defined by

$$Y(h) = p_e h = hK(1 - \frac{h}{r}).$$

Maximizing in the usual way through differentiation, we have

$$Y'(h) = K(1 - \frac{h}{r}) - \frac{hK}{r} = 0 \Rightarrow h = \frac{r}{2}.$$

For this rate, our harvest is $\frac{r}{2}K(1 - \frac{r/2}{r}) = \frac{rK}{4}$, and the fish population approaches its equilibrium point $K(1 - \frac{r/2}{r}) = \frac{K}{2}$. △

2.9 Nonlinear Stability Theorems

In this section, we state two important theorems regarding the nonlinear terms that are dropped off in linearization.

Theorem 2.16 (Poincare–Perron). *For the ODE system*

$$\frac{d\vec{y}}{dt} = \vec{f}(\vec{y}),$$

suppose $\vec{y}_e$ denotes an equilibrium point and that $\vec{f}$ is twice continuously differentiable for $\vec{y}$ in a neighborhood of $\vec{y}_e$. (I.e., all second order partial derivatives of each component of $\vec{f}$ are continuous.) Then $\vec{y}_e$ is stable or unstable as follows:

1. *If the eigenvalues of $\vec{f}'(\vec{y}_e)$ all have negative real part, then $\vec{y}_e$ is asymptotically stable.*

2. *If any of the eigenvalues of $\vec{f}'(\vec{y}_e)$ has positive real part then $\vec{y}_e$ is unstable.*

Example 2.10. Show that the assumptions of the Poincare–Perron Theorem are satisfied for the competition model studied in Example 2.8, and consequently that for cases for which none of the eigenvalues has 0 real part asymptotic stability or instability can be concluded from linear stability or instability.

For the competition model studied in Example 2.6, we have

$$\begin{aligned} f_1(y_1, y_2) =& r_1 y_1 (1 - \frac{y_1 + s_1 y_2}{K_1}) \\ f_2(y_1, y_2) =& r_2 y_2 (1 - \frac{y_2 + s_2 y_1}{K_2}), \end{aligned}$$

and consequently

$$\frac{\partial^2 f_1}{\partial y_1^2}(y_1, y_2) = -2\frac{r_1}{K_1}; \quad \frac{\partial^2 f_1}{\partial y_2^2}(y_1, y_2) = 0; \quad \frac{\partial^2 f_1}{\partial y_1 \partial y_2}(y_1, y_2) = \frac{\partial^2 f_1}{\partial y_2 \partial y_1}(y_1, y_2) = -\frac{r_1 s_1}{K_1},$$

and

$$\frac{\partial^2 f_2}{\partial y_1^2}(y_1, y_2) = 0; \quad \frac{\partial^2 f_2}{\partial y_2^2}(y_1, y_2) = -2\frac{r_2}{K_2}; \quad \frac{\partial^2 f_2}{\partial y_1 \partial y_2}(y_1, y_2) = \frac{\partial^2 f_2}{\partial y_2 \partial y_1}(y_1, y_2) = -\frac{r_2 s_2}{K_2}.$$

Since each of these eight second partials is continuous the assumptions of the Poincare–Perron theorem are satisfied. △

Remark. This is not the most general statement of the Poincare–Perron theorem, but the assumption of differentiability usually holds true, and it is easy to check in practice.

Before proving the Poincare–Perron Theorem, we state a more abstract theorem that gives more information. The formulation I'll give is a slight generalization of the original 1960 result of Hartman [Evans]. In stating this result, we will adopt the following convention: we will use the notation $\vec{y}(t; \vec{y}_0)$ to denote the solution of the initial value problem $\vec{y}' = \vec{f}(\vec{y})$, $\vec{y}(0) = \vec{y}_0$. The *flow* of this differential equation is defined as the operator

$$S_t \vec{y}_0 = \vec{y}(t; \vec{y}_0).$$

Theorem 2.17 (Hartman–Grobman). *For the ODE system*

$$\frac{d\vec{y}}{dt} = \vec{f}(\vec{y}); \quad \vec{y}(0) = \vec{y}_0,$$

suppose $\vec{y}_e$ is an equilibrium point for which $\vec{f}'(\vec{y}_e)$ has no eigenvalues with real part 0, and that $\vec{f}$ is twice continuously differentiable in a neighborhood of $\vec{y}_e$. Then solutions $\vec{y}(t; \vec{y}_0)$ are similar to solutions $\vec{z}(t; \vec{z}_0)$ of the linear problem

$$\frac{d\vec{z}}{dt} = \vec{f}'(\vec{y}_e)\vec{z}; \quad \vec{z}(0) = \vec{z}_0$$

in the following sense: there exists a differentiable vector function $\vec{\phi}$ with a differentiable inverse $\vec{\phi}^{-1}$ (technically, $\vec{\phi}$ is a diffeomorphism*) so that $\vec{\phi}$ maps neighborhoods of 0 into neighborhoods of $\vec{y}_e$ and*

$$\vec{y}(t; \vec{\phi}(\vec{z}_0)) = \vec{\phi}(\vec{z}(t; \vec{z}_0)).$$

In order to better understand exactly what this theorem is telling us, let's consider a system of two equations for which $\vec{f}'(\vec{y}_e)$ has one eigenvalue with positive real part and one eigenvalue with negative real part (i.e., a saddle). We know from our study of linear systems that the phase plane for the linear system contains two straight lines, one along which trajectories move toward $\vec{y}_e$ and one along which trajectories move away from $\vec{y}_e$. The function $\vec{\phi}$ provides a map from this configuration to a similar configuration in the phase plane for the nonlinear system. Perhaps the easiest way to see this is to consider a vector point $\vec{u}_0$ on the line with an out-going trajectory, and to study the effect of $\vec{\phi}$ on the line between 0 and $\vec{u}_0$. In particular, since $\vec{\phi}$ is invertible, this line must be mapped into a curve in the $\vec{y}$ phase plane connecting $\phi(0) = \vec{y}_e$ to $\vec{\phi}(\vec{u}_0)$. (See Figure 2.14.) In this way, we see that the most important aspects of the phase plane, at least with regard to stability,

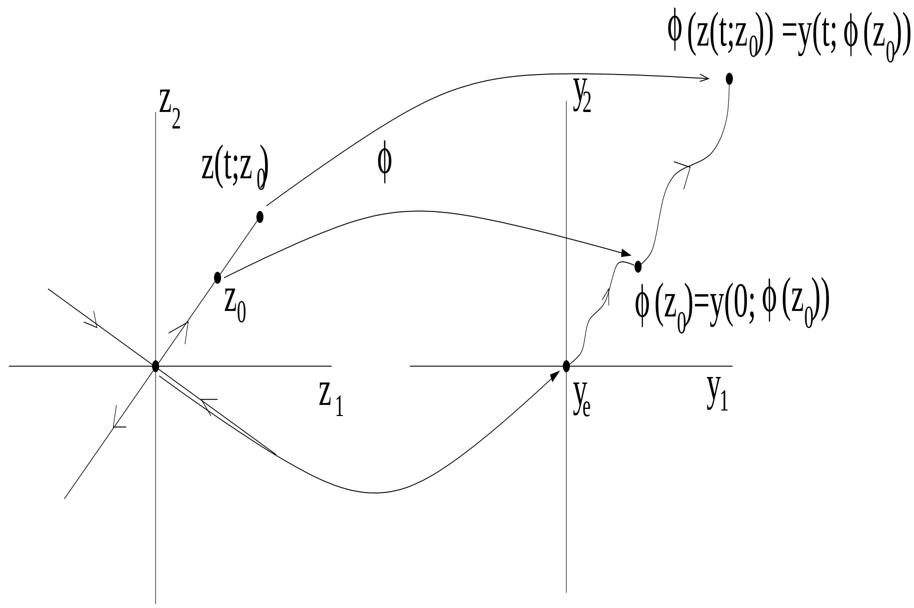

Figure 2.14: The Hartman–Grobman mapping from a linear phase phase space to a nonlinear phase space.

are preserved under $\vec{\phi}$. More precisely, all outgoing trajectories remain outgoing under $\vec{\phi}$ and all incoming trajectories remain incoming under $\vec{\phi}$. Thus either asymptotic stability or instability is preserved.

Example 2.11. Explain how the Hartman–Grobman theorem can be interpreted in the case of the equilibrium point $(0,0)$ for the competition model of Example 2.8.

We have shown in Example 2.8 that for this equilbrium point the Jacobian matrix has two positive eigenvalues so that the linear system has an unstable node at $\vec{z}_e = \vec{0}$. This means that at the linear level all trajectories eventually flow away from the equilibrium point. (Even in this case, due to possible cancellation, trajectories may transiently move toward the equilibrium point.) The Hartman–Grobman theorem asserts that we can conclude that all trajectories associated with the nonlinear ODE also eventually flow away from the equilibrium point. Note particularly that this is more information than we obtained for the same equilibrium point from Poincare–Perron. △

2.10 Solving ODE Systems with matrix exponentiation

In this section we introduce a new method for solving linear systems of ODE that will allow us to prove the Poincare–Perron Theorem. We begin by writing the system

$$\begin{aligned}\frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= \frac{g}{l} y_1,\end{aligned}$$

in matrix form

$$\vec{y}' = A\vec{y}, \tag{2.17}$$

where

$$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix}.$$

If we define matrix exponentiation through Taylor expansion,

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + ...,$$

then as in the case with single equations, we can conclude

$$y(t) = e^{At}\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}$$

is a solution to (2.17). (This assertion can be checked through direct term-by-term differentiation.) In the event that A is diagonal (which is not the case in our example), e^{At} is straightforward to evaluate. For

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix},$$

we have

$$\begin{aligned}
e^{At} =& I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + ... \\
=& \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} t^2 \\
&+ \frac{1}{6}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + ... \\
=& \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} t + \frac{1}{2}\begin{pmatrix} a_1^2 & 0 \\ 0 & a_2^2 \end{pmatrix} t^2 + \frac{1}{6}\begin{pmatrix} a_1^3 & 0 \\ 0 & a_2^3 \end{pmatrix} + ... \\
=& \begin{pmatrix} e^{a_1t} & 0 \\ 0 & e^{a_2t} \end{pmatrix}.
\end{aligned}$$

In the event that A is not diagonal, we will proceed by choosing a change of basis that diagonalizes A. A general procedure for diagonalizing a matrix is outlined in the following three steps.

1. For an $n \times n$ matrix A, find n linearly independent eigenvectors of A, $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$.

2. Form a matrix P that consists of $\vec{v}_1$ as its first column, $\vec{v}_2$ as its second column, etc., with finally $\vec{v}_n$ as its last column.

3. The matrix $P^{-1}AP$ will then be diagonal with diagonal entries the eigenvalues associated with $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$: λ_1, λ_2, ..., λ_n.

Remark on Steps 1–3. First, it is not always the case that a matrix will have n linearly independent eigenvectors, and in situations for which this is not the case, more work is required (in particular, instead of diagonalizing the matrix, we put it in Jordon canonical form). Under the assumption that Step 1 is possible, the validity of Steps 2 and 3 is straightforward. If P is the matrix of eigenvectors, then

$$AP = (\lambda_1 \vec{v}_1 \vdots \lambda_2 \vec{v}_2 \vdots \dots \vdots \lambda_n \vec{v}_n);$$

that is, the matrix containing as its k^{th} column the vector $\lambda_k \vec{v}_k$. Multiplying on the left by P^{-1}, which must exist if the $\vec{v}_k$ are all linearly independent, we have

$$P^{-1}(\mu_1 \vec{v}_1 \vdots \mu_2 \vec{v}_2 \vdots \dots \vdots \mu_n \vec{v}_n) = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mu_n \end{pmatrix}.$$

In this last calculation, we are almost computing $P^{-1}P$, which would yield the identity matrix.

Example 2.12. Diagonalize the matrix

$$A = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix},$$

which arose upon linearization of the pendulum equation about the equilibrium point $(\pi, 0)$.

We have already seen in Example 3.5 that the eigenvalues and eigenvectors for this equation are

$$\lambda_1 = -\sqrt{\frac{g}{l}}, \vec{v}_1 = \begin{pmatrix} 1 \\ -\sqrt{\frac{g}{l}} \end{pmatrix}; \quad \lambda_2 = \sqrt{\frac{g}{l}}, \vec{v}_2 = \begin{pmatrix} 1 \\ \sqrt{\frac{g}{l}} \end{pmatrix}.$$

Accordingly, we define

$$P = \begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{g}{l}} & \sqrt{\frac{g}{l}} \end{pmatrix} \Rightarrow P^{-1} = \sqrt{\frac{l}{4g}} \begin{pmatrix} \sqrt{\frac{g}{l}} & -1 \\ \sqrt{\frac{g}{l}} & 1 \end{pmatrix},$$

with which we compute

$$\begin{aligned} D =& P^{-1}AP = \sqrt{\frac{l}{4g}} \begin{pmatrix} \sqrt{\frac{g}{l}} & -1 \\ \sqrt{\frac{g}{l}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{g}{l}} & \sqrt{\frac{g}{l}} \end{pmatrix} \\ =& \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{\frac{l}{g}} \\ \frac{1}{2} & \frac{1}{2}\sqrt{\frac{l}{g}} \end{pmatrix} \begin{pmatrix} -\sqrt{\frac{g}{l}} & \sqrt{\frac{g}{l}} \\ \frac{g}{l} & \frac{g}{l} \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{g}{l}} & 0 \\ 0 & +\sqrt{\frac{g}{l}} \end{pmatrix}. \end{aligned}$$

Note particularly that the order of eigenvalues down the diagonal agrees with the order in which the eigenvectors appear in our construction of P. △

Once the diagonalizing matrix P has been constructed, it can be used in computing e^{At}. In the event that $A = PDP^{-1}$, we have

$$\begin{aligned}
e^{At} &= I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + ...,\\
&= I + (PDP^{-1})t + \frac{1}{2}(PDP^{-1})^2 + \frac{1}{3!}(PDP^{-1})^3 + ...\\
&= I + (PDP^{-1})t + \frac{1}{2}(PDP^{-1}PDP^{-1}) + \frac{1}{3!}(PDP^{-1}PDP^{-1}PDP^{-1}) + ...\\
&= P(I + Dt + \frac{1}{2}D^2 + \frac{1}{3!}D^3 + ...)P^{-1}\\
&= Pe^{Dt}P^{-1}.
\end{aligned}$$

We are now in a position to very quickly write down the solution to a fairly general linear system of ODE with constant coefficients.

Theorem 2.18. *For the ODE system*

$$\vec{y}' = A\vec{y}; \quad \vec{y}(0) = \vec{y}_0,$$

suppose A is a constant $n \times n$ matrix with n linearly independent eigenvectors $\{\vec{v}_k\}_{k=1}^n$, and a matrix P is constructed by using $\vec{v}_1$ as the first column, $\vec{v}_2$ as the second column etc. Then the ODE system is solved by

$$\vec{y}(t) = e^{At}\vec{y}_0 = Pe^{Dt}P^{-1}\vec{y}_0,$$

where D is a diagonal matrix with the eigenvalues of A on the diagonal.

Example 2.13. Use the method of matrix exponentiation to solve the linear ODE $\vec{y}' = A\vec{y}$, where

$$A = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & 0 \end{pmatrix}.$$

In this case, we computed P and P^{-1} in Example 3.11, so we can immediately write

$$\begin{aligned}
\vec{y}(t) &= e^{At}\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = Pe^{Dt}P^{-1}\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}\\
&= \begin{pmatrix} 1 & 1 \\ -\sqrt{\frac{g}{l}} & \sqrt{\frac{g}{l}} \end{pmatrix}\begin{pmatrix} e^{-\sqrt{\frac{g}{l}}t} & 0 \\ 0 & e^{\sqrt{\frac{g}{l}}t} \end{pmatrix} \cdot (\sqrt{\frac{l}{4g}})\begin{pmatrix} \sqrt{\frac{g}{l}} & -1 \\ \sqrt{\frac{g}{l}} & 1 \end{pmatrix}\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}\\
&= \begin{pmatrix} e^{-\sqrt{\frac{g}{l}}t} & e^{\sqrt{\frac{g}{l}}t} \\ -\sqrt{\frac{g}{l}}e^{-\sqrt{\frac{g}{l}}t} & \sqrt{\frac{g}{l}}e^{\sqrt{\frac{g}{l}}t} \end{pmatrix}\begin{pmatrix} \frac{1}{2}y_1(0) - \frac{1}{2}\sqrt{\frac{l}{g}}y_2(0) \\ \frac{1}{2}y_1(0) + \frac{1}{2}\sqrt{\frac{l}{g}}y_2(0) \end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{2}(y_1(0) - \sqrt{\frac{l}{g}}y_2(0))e^{-\sqrt{\frac{g}{l}}t} + \frac{1}{2}(y_1(0) + \sqrt{\frac{l}{g}}y_2(0))e^{\sqrt{\frac{g}{l}}t} \\ -\frac{1}{2}(y_1(0) - \sqrt{\frac{l}{g}}y_2(0))\sqrt{\frac{g}{l}}e^{-\sqrt{\frac{g}{l}}t} + \frac{1}{2}(y_1(0) + \sqrt{\frac{l}{g}}y_2(0))\sqrt{\frac{g}{l}}e^{\sqrt{\frac{g}{l}}t} \end{pmatrix}.
\end{aligned}$$

This final expression should be compared with our previous answer to the same problem, obtained in Example 3.5,

$$\vec{y}(t) = \frac{1}{2}\Big(y_1(0) - \sqrt{\frac{l}{g}}y_2(0)\Big)\begin{pmatrix} 1 \\ -\sqrt{\frac{g}{l}} \end{pmatrix} e^{-\sqrt{\frac{g}{l}}t} + \frac{1}{2}\Big(y_1(0) + \sqrt{\frac{l}{g}}y_2(0)\Big)\begin{pmatrix} 1 \\ \sqrt{\frac{g}{l}} \end{pmatrix} e^{\sqrt{\frac{g}{l}}t}.$$

△

2.11 Taylor's Formula for Functions of Multiple Variables

The proof of the Poincare–Perron Theorem requires a Taylor expansion of $\vec{f}(\vec{y})$ about the point $\vec{y}_e$; i.e., a Taylor expansion of a vector function about a vector variable. First, we state Taylor's theorem for a single function of a vector variable.

Theorem 2.19. *Suppose a function $f(y_1, y_2, \ldots, y_n)$ and all its partial derivatives up to order $N+1$ are continuous in a neighborhood of the point $(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)$. Then for each point $(y_1, y_2, \ldots, y_n)$ in this neighborhood, there exists a value $\theta \in (0,1)$ so that ,*

$$\begin{aligned} f(y_1, y_2, \ldots, y_n) = & \sum_{k=0}^{n} \frac{1}{k!}\Big((y_1 - \hat{y}_1)\frac{\partial}{\partial y_1} + (y_2 - \hat{y}_2)\frac{\partial}{\partial y_2} + \cdots + (y_n - \hat{y}_n)\frac{\partial}{\partial y_2}\Big)^k f \\ & + \frac{1}{(n+1)!}\Big((y_1 - \hat{y}_1)\frac{\partial}{\partial y_1} + (y_2 - \hat{y}_2)\frac{\partial}{\partial y_2} + \cdots + (y_n - \hat{y}_n)\frac{\partial}{\partial y_2}\Big)^{n+1} f, \end{aligned}$$

where the summed expression is evaluated at the point $(\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)$, while the final error expression is evaluated at

$$(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n) + \theta(y_1 - \hat{y}_1, y_2 - \hat{y}_2, \ldots, y_n - \hat{y}_n).$$

We observe that the expression for f in this theorem can be written more succinctly in the vector form

$$f(\vec{y}) = \sum_{k=0}^{n} \frac{1}{k!}\Big((\vec{y} - \vec{y}_e)\cdot\nabla\Big)^k f(\vec{y}_e) + \frac{1}{(n+1)!}\Big((\vec{y} - \vec{y}_e)\cdot\nabla\Big)^{n+1} f(\bar{y}),$$

where $\vec{y}_e = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_n)$ and

$$\bar{y} = \vec{y}_e + \theta(\vec{y} - \vec{y}_e).$$

In writing a Taylor expansion for a vector function $\vec{f}(\vec{y})$, we can proceed by using the above theorem to expand each component of f. By doing this, we can prove the following theorem.

Theorem 2.20. *Suppose $\vec{f}(\vec{y})$ is twice continuously differentiable in a neighborhood of the point $\vec{y}_e$, and let $\bar{B}_r(\vec{y}_e)$ denote a closed ball of radius r contained in this neighborhood. Then there exists a function $\vec{Q} = \vec{Q}(\vec{y}_e, \vec{y})$ so that for all $\vec{y} \in \bar{B}_r(\vec{y}_e)$*

$$\vec{f}(\vec{y}) = \vec{f}(\vec{y}_e) + \vec{f}'(\vec{y}_e)(\vec{y} - \vec{y}_e) + \vec{Q}(\vec{y}_e, \vec{y}),$$

with

$$|\vec{Q}(\vec{y}_e, \vec{y})| \leq C|\vec{y} - \vec{y}_e|^2,$$

where C depends only on $\vec{f}$, and $\vec{y}_e$.

2.12 Proof of Part 1 of Poincare–Perron

In this section, we prove Part 1 of the Poincare–Perron Theorem, stated in Subsection 2.9.

We begin with the equation $\vec{y}' = \vec{f}(\vec{y})$, for which $\vec{y}_e$ is assumed to be an equilibrium point. Under our assumption that $\vec{f}(\vec{y}_e)$ is twice continuously differeniable in a neighborhood of $\vec{y}_e$, we can write the Taylor expansion

$$\vec{f}(\vec{y}) = \vec{f}(\vec{y}_e) + \vec{f}'(\vec{y}_e)(\vec{y} - \vec{y}_e) + \vec{Q}(\vec{y}, \vec{y}_e),$$

where we know that $\vec{f}(\vec{y}_e) = 0$ and $|\vec{Q}(\vec{y}_e, \vec{y})| \le C|\vec{y} - \vec{y}_e|^2$. Upon linearization with $\vec{z} = \vec{y} - \vec{y}_e$, we obtain

$$\vec{z}' = \vec{f}'(\vec{y}_e)\vec{z} + \vec{Q}(\vec{y}, \vec{y}_e),$$

where $|\vec{Q}(\vec{y}_e, \vec{y})| \le C|\vec{z}|^2$. We can re-write this equation for $\vec{z}$ as

$$(e^{-\vec{f}'(\vec{y}_e)t}\vec{z})' = e^{-\vec{f}'(\vec{y}_e)t}\vec{Q}(\vec{y}, \vec{y}_e),$$

which can be integrated from 0 to t to give

$$\vec{z}(t) = e^{\vec{f}'(\vec{y}_e)t}\vec{z}(0) + \int_0^t e^{\vec{f}'(\vec{y}_e)(t-s)}\vec{Q}(\vec{y}(s), \vec{y}_e)ds.$$

In the first case of the theorem, the eigenvalues of $\vec{f}'(\vec{y}_e)$ all have negative real part. If we let λ_R denote the real part of the largest eigenvalue of $\vec{f}'(\vec{y}_e)$, then there exists some constant C_1 so that the largest entry in the matrix $e^{\vec{f}'(\vec{y}_e)(t-s)}$ is bounded in complex modulus by $C_1 e^{\lambda_R t}$. In this way, we see that

$$|\vec{z}(t)| \le K_1 e^{\lambda_R t}|\vec{z}(0)| + K_2 \int_0^t e^{\lambda_R(t-s)}|\vec{z}(s)|^2 ds, \tag{2.18}$$

where K_1 and K_2 are positive constants. Let $\zeta(t)$ be defined as follows:

$$\zeta(t) := \sup_{s \in [0,t]} |\vec{z}(s)e^{-\lambda_R s}|,$$

and note that (2.18) can be rearranged as

$$|\vec{z}(t)e^{-\lambda_R t}| \le K_1|\vec{z}(0)| + K_2 \int_0^t e^{-\lambda_R s}|\vec{z}(s)|^2 ds.$$

Clearly, since the right-hand side of (2.18) increases with t, we can take a sup norm on each side of this last expression to obtain

$$\begin{aligned}
\zeta(t) \le& K_1|\vec{z}(0)| + K_2 \int_0^t e^{-\lambda_R s}|\vec{z}(s)|^2 ds. \\
&\le K_1\zeta(0) + K_2\zeta(t)^2 \int_0^t e^{\lambda_R s} ds \\
&\le K_1\zeta(0) + \frac{K_2}{|\lambda_R|}\zeta(t)^2.
\end{aligned} \tag{2.19}$$

Note especially that since $\zeta(0) = |\vec{z}(0)|$, we are justified in choosing it as small as we like. (In our definition of stability, we take $|\vec{z}(0)| \le \delta$, where δ may be chosen.) The following argument is referred to as *continuous induction*. Since it is important in its own right, we separate it out as a lemma.

Lemma 2.21 (continuous induction). *Suppose there exists a constant C so that*

$$\zeta(t) \le C(\zeta(0) + \zeta(t)^2), \tag{2.20}$$

for all $t \ge 0$. If $\zeta(0) < \min\{1, 1/(4C^2)\}$, then

$$\zeta(t) < 2C\zeta(0), \tag{2.21}$$

for all $t \ge 0$.

Proof of the claim. We first observe that by (2.20), and for $\zeta(0) < 1$,

$$\zeta(0) \le C(\zeta(0) + \zeta(0)^2) < C(\zeta(0) + \zeta(0)) = 2C\zeta(0),$$

so (2.21) is satisfied for $t = 0$ and, by continuity for some interval $t \in [0, T]$. Let T, if it exists, denote the first time for which $\zeta(T) = 2C\zeta(0)$. Then

$$\zeta(T) \le C(\zeta(0) + \zeta(T)^2) = C(\zeta(0) + 4C^2\zeta(0)^2) < C(\zeta(0) + \zeta(0)) = 2C\zeta(0),$$

which contradicts the existence of T. If no such T exists, (2.21) must be true for all t. $\square$

We now conclude our proof of Poincare–Perron as follows. Let $C = \max\{K_1, \frac{K_2}{|\lambda_R|}\}$. Then by (2.19), we have

$$\zeta(t) \le C(\zeta(0) + \zeta(t)^2),$$

where since $\zeta(0)$ can be chosen as small as we like, we can take $\zeta(0) < \min\{1, 1/(4C^2)\}$, so that the assumptions of our lemma on continuous induction are met. We conclude

$$\zeta(t) < 2C\zeta(0),$$

or

$$\sup_{s \in [0,t]} |\vec{z}(s) e^{-\lambda_R s}| < 2C\zeta(0).$$

In particular,

$$|\vec{z}(t) e^{-\lambda_R t}| < 2C\zeta(0),$$

giving at last

$$|\vec{z}(t)| \le 2C\zeta(0) e^{\lambda_R t}.$$

Since $\lambda_R < 0$,

$$\lim_{t\to\infty} |\vec{z}(t)| = \lim_{t\to\infty} 2C\zeta(0) e^{\lambda_R t} = 0,$$

which means, according to our definition of asymptotic stability that $\vec{y}_e$ is asymptotically stable. $\square$

3 Uniqueness Theory

Example 3.1. Consider the following problem: Ignoring air resistance, determine an exact form for the time at which an object lauched vertically from a height h with velocity v strikes the earth. According to Newton's second law of motion, the height of the object $y(t)$ can be described through,

$$y(t) = -g\frac{t^2}{2} + vt + h.$$

Setting $y(t) = 0$, we find,

$$-gt^2/(2h) + vt/h + 1 = 0,$$

with solution,

$$t = \frac{-v \pm \sqrt{v^2 + 2gh}}{-g}.$$

While we know that there is only one time at which the object can strike the ground, our model gives us two different times. This is a problem of uniqueness. (In this case, the resolution is straightforward: taking $-$ makes $t > 0$ and corresponds with the time we are looking for; taking $+$ makes $t < 0$ and corresponds with the object's trajectory being traced backward in time along its parabolic arc to the ground.) △

Though the question of uniqueness arises in every type of equation—algebraic, differential, integral, integrodifferential, stochastic, etc.—we will only develop a (relatively) full theory in the case of ordinary differential equations.

Example 3.2. Consider the ordinary differential equation,

$$\frac{dy}{dt} = y^{2/3}; \quad y(0) = 0.$$

Solving by separation of variables, we find $y(t) = t^3/27$, which we compare with the following MATLAB script:

```
>>[t,y]=ode23(inline('y^(2/3)','t','y'),[0 ,.5],0)
t =
0
0.0500
0.1000
0.1500
0.2000
0.2500
0.3000
0.3500
0.4000
0.4500
0.5000
y =
0
```

0
0
0
0
0
0
0
0
0
0

According to MATLAB, the solution is $y(t) = 0$ for all t, and indeed it is straightforward to check that this is a valid solution to the equation. We find, in fact, that for any $c > 0$, the function $y(t)$ given as

$$y(t) = \begin{cases} \frac{(t-c)^3}{27}, & t \geq c \\ 0, & t \leq c \end{cases}$$

satisfies this equation. In practice, this is the fundamental issue with uniqueness: If our model does not have a unique solution, we don't know whether or not the solution MATLAB (or alternative software) gives us is the one that corresponds with the phenomenon we're modeling. △

Two critical questions are apparent: 1. When can we insure that this problem won't arise (that solutions are unique)? and 2. In the case of nonuniqueness, can we develop a theory that selects the correct solution? The second of these questions can only be answered in the context of the phenomenon we're modeling. For example, in Example 3.5, we selected $t > 0$ because we were trying to predict a future time, and only one solution satisfied $t > 0$. As we observed, however, the other solution answered a different question that might have been posed: how long ago would the object have had to leave the ground to get to height h? Fortunately, for the first of our two questions—at least in the case of ODE—we have a definitive general theorem.

Theorem 3.1. *(ODE Uniqueness) Let $f(t, y) = (f_1(t, y), f_2(t, y), ..., f_n(t, y))^{tr}$ be a vector function whose components are each continuous in both t and y in some neighborhood $a \leq t \leq b$ and $a_1 \leq y_1 \leq b_1$,$a_2 \leq y_2 \leq b_2$,...,$a_n \leq y_n \leq b_n$ and whose partial derivatives $\partial_{y_l} f_k(t, y)$ are continuous in both t and y in the same neighborhoods for each $l, k = 1, ..., n$. Then given any initial point $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$ such that $a < t_0 < b$ and $a_k < y_{0_k} < b_k$ for all $k = 1, ...n$, any solution to*

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0$$

is unique on the neighborhood of continuity.

Example 3.2 continued. Notice that our equation from Example 3.2 better not satisfy the conditions of Theorem 3.1. In this case, $f(t, y) = y^{2/3}$, which is continuous in both t (trivially) and y. Computing $\partial_y f(t, y) = \frac{2}{3} y^{-1/3}$, we see that the y-derivative of f is not continuous at the initial value $y = 0$. △

Example 3.3. Consider again the Lotka–Volterra predator–prey model, which we can rewrite in the notation of Theorem 3.1 as ($y_1 = x$, $y_2 = y$),

$$\begin{aligned}\frac{dy_1}{dt} &= ay_1 - by_1y_2; \quad y_1(t_0) = y_{0_1}\\ \frac{dy_2}{dt} &= -ry_2 + cy_1y_2; \quad y_2(t_0) = y_{0_2}.\end{aligned}$$

In this case, the vector $f(t, y)$ is

$$\begin{pmatrix} f_1(t, y_1, y_2) \\ f_2(t, y_1, y_2) \end{pmatrix} = \begin{pmatrix} ay_1 - by_1y_2 \\ -ry_2 + cy_1y_2 \end{pmatrix}.$$

As polynomials, $f_1, f_2, \partial_{y_1} f_1, \partial_{y_2} f_1, \partial_{y_1} f_2$, and $\partial_{y_2} f_2$ must all be continuous for all t, y_1, and y_2, so any solution we find to these equations must be unique. △

Idea of the uniqueness proof. Before proceeding with a general proof of Theorem 3.1, we will work through the idea of the proof in the case of a concrete example. Consider the ODE

$$\frac{dy}{dt} = y^2; \quad y(0) = 1, \tag{3.1}$$

and suppose we want to establish uniqueness on the intervals $a \le t \le b$ and $a_1 \le y \le b_1$, with $0 \in (a, b)$ and $1 \in (a_1, b_1)$. We begin by supposing that $y_1(t)$ and $y_2(t)$ are both solutions to (3.1) and defining the squared difference between them as a variable,

$$E(t) := (y_1(t) - y_2(t))^2.$$

Our goal becomes to show that $E(t) \equiv 0$; that is, that $y_1(t)$ and $y_2(t)$ must necessarily be the same function. Computing directly, we have

$$\begin{aligned}\frac{dE}{dt} &= 2(y_1(t) - y_2(t))(\frac{dy_1}{dt} - \frac{dy_2}{dt})\\ &= 2(y_1(t) - y_2(t))(y_1(t)^2 - y_2(t)^2)\\ &= 2(y_1(t) - y_2(t))(y_1(t) - y_2(t))(y_1(t) + y_2(t))\\ &= 2(y_1(t) - y_2(t))^2(y_1(t) + y_2(t))\\ &= 2E(t)(y_1(t) + y_2(t)).\end{aligned}$$

Since y_1 and y_2 are both assumed less than b_1, we conclude the differential inequality

$$\frac{dE}{dt} \le 2E(t)(2b_1),$$

which upon multiplication by the (non-negative) *integrating factor* e^{-4b_1t} can be written as

$$\frac{d}{dt}[e^{-4b_1t}E(t)] \le 0.$$

Integrating, we have

$$\int_0^t \frac{d}{dt}[e^{-4b_1s}E(s)]ds = e^{-4b_1s}E(s)\Big|_0^t = e^{-4b_1t}E(t) - E(0) \le 0.$$

Recalling that $y_1(0) = y_2(0) = 1$, we observe that $E(0) = 0$ and consequently $E(t) \leq 0$. But $E(t) \geq 0$ by definition, so that we can conclude that $E(t) = 0$. $\square$

Proof of Theorem 3.1. In order to restrict the tools of this proof to a theorem that should be familiar to most students, we will carry it out only in the case of a single equation. The extension to systems is almost identical, only requiring a more general form of the Mean Value Theorem.

We begin as before by letting $y_1(t)$ and $y_2(t)$ represent two solutions of the ODE

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0.$$

Again, we define the squared difference between $y_1(t)$ and $y_2(t)$ as $E(t) := (y_1(t) - y_2(t))^2$. Computing directly, we have now

$$\begin{aligned}\frac{dE}{dt} &= 2(y_1(t) - y_2(t))(\frac{dy_1}{dt} - \frac{dy_2}{dt}) \\ &= 2(y_1(t) - y_2(t))(f(t, y_1) - f(t, y_2)).\end{aligned}$$

At this point, we need to employ the Mean Value Theorem (see Appendix A), which asserts in this context that for each t there exists some number $c \in [y_1, y_2]$ so that

$$f'(c) = \frac{f(t, y_1) - f(t, y_2)}{y_1 - y_2}, \quad \text{or} \quad f(t, y_1) - f(t, y_2) = \partial_y f(t, c)(y_1 - y_2).$$

Since $\partial_y f$ is assumed continuous on the closed interval $t \in [a, b], y \in [a_1, b_1]$, the Extreme Value Theorem (see Appendix A) guarantees the existence of some constant L so that $|\partial_y f(t, y)| \leq L$ for all $t \in [a, b], y \in [a_1, b_1]$. We have, then, the so-called *Lipschitz inequality*,

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|. \tag{3.2}$$

We conclude that

$$\frac{dE}{dt} \leq 2|y_1(t) - y_2(t)|L|y_1(t) - y_2(t)| = 2LE(t),$$

from which we conclude exactly as above that $E(t) \equiv 0$. $\square$

Remark on the Lipschitz Inequality. Often the ODE uniqueness theorem is stated under the assumption of the Lipschitz inequality (3.2). I have chosen to state it here in terms of the continuity of derivatives of f because that is typically an easier condition to check. Since continuity of derivatives implies the Lipschitz inequality, the Lipschitz formulation is more general.

4 Existence Theory

Existence theory is one of the most abstract topics in applied mathematics. The idea is to determine that a solution to some problem exists, even if the solution cannot be found.

Example 4.1. Prove that there exists a real solution to the algebraic equation

$$x^7 + 6x^4 + 3x + 9 = 0.$$

While actually finding a real solution to this equation is quite difficult, it's fairly easy to recognize that such a solution must exist. As x goes to $+\infty$, the left hand side becomes positive, while as x goes to $-\infty$ the left hand side becomes negative. Somewhere in between these two extremes, the left hand side must equal 0. In this way we have deduced that a solution exists without saying much of anything about the nature of the solution. (Mathematicians in general are notorious for doing just this sort of thing.) $\triangle$

If we really wanted to ruin MATLAB's day, we could assign it the ODE

$$\frac{dy}{dt} = t^{-1}; \quad y(0) = 1.$$

Solving by direct integration, we see that $y(t) = \log t + C$, so that no value of C can match our initial data. (The current version of MATLAB simply crashes.) As with the case of uniqueness, we would like to insure the existence of some solution before trying to solve the equation. Fortunately, we have the following theorem, due to Picard.

Theorem 4.1. *(ODE Existence)*[2] *Let $f(t, y) = (f_1(t, y), f_2(t, y), ..., f_n(t, y))^{tr}$ be a vector function whose components are each continuous in both t and y in some neighborhood $a \le t \le b$ and $a_1 \le y_1 \le b_1$,$a_2 \le y_2 \le b_2$,...,$a_n \le y_n \le b_n$ and whose partial derivatives $\partial_{y_l} f_k(t, y)$ are continuous in both t and y in the same neighborhoods for each $l, k = 1, ..., n$. Then given any initial point $(t_0, y_0) \in \mathbb{R} \times \mathbb{R}^n$ such that $a < t_0 < b$ and $a_k < y_{0_k} < b_k$ for all $k = 1, ...n$, there exists a solution to the ODE*

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0 \tag{4.1}$$

for some domain $|t - t_0| < \tau$, where $\tau > 0$ may be extremely small. Moreover, the solution y is a continuous function of the independent variable t and of the parameters t_0 and y_0.

Example 4.2. Consider the ODE

$$\frac{dy}{dt} = y^2; \quad y(0) = 1.$$

Since $f(t, y) = y^2$ is clearly continuous with continuous derivatives, Theorem 3.3 guarantees that a solution to this ODE exists. Notice particularly, however, that the interval of existence is not specified. To see exactly what this means, we solve the equation by separation of variables, to find

$$y(t) = \frac{1}{1 - t},$$

from which we observe that though $f(y)$ and its derivatives are continuous for all t and y, existence is lost at $t = 1$. Referring to the statement of our theorem, we see that this statement is equivalent to saying that $\tau = 1$. Unfortunately, our general theorem does not specify τ for us a priori. $\triangle$

Idea of the proof of Theorem 4.1, single equations. Consider the ODE

$$\frac{dy}{dt} = y; \quad y(0) = 1.$$

[2] The assumptions here are exactly the same as those for Theorem 3.2, so together Theorems 3.2 and 3.3 constitute a complete existence–uniqueness theory.

Our goal here is to establish that a solution exists without every actually finding the solution. (Though if we accidentally stumble across a solution on our way, that's fine too.) We begin by simply integrating both sides, to obtain the *integral equation*

$$y(t) = 1 + \int_0^t y(s)ds.$$

(Unlike in the method of separation of variables, we have integrated both sides with respect to the same variable, t.) Next, we try to find a solution by an iteration. (Technically, *Picard Iteration.*) The idea here is that we guess at a solution, say $y_{\text{guess}}(t)$ and then use our integral equation to (hopefully) improve our guess through the calculation

$$y_{\text{new guess}}(t) = 1 + \int_0^t y_{\text{old guess}}(s)ds.$$

Typically, we call our first guess $y_0(t)$ and use the initial value: here, $y_0(t) = 1$. Our second guess, $y_1(t)$, becomes

$$y_1(t) = 1 + \int_0^t y_0(s)ds = 1 + \int_0^t 1ds = 1 + t.$$

Similiarly, we compute our next guess (iteration),

$$y_2(t) = 1 + \int_0^t y_1(s)ds = 1 + \int_0^t (1+s)ds = 1 + t + \frac{t^2}{2}.$$

Proceeding similarly, we find that

$$y_n(t) = \sum_{k=0}^{n} \frac{t^k}{k!} \Rightarrow \lim_{n\to\infty} y_n(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!},$$

and our candidate for a solution becomes $y(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}$, an infinite series amenable to such tests as the *integral test*, the *comparison test*, the *limit comparison test*, the *alternating series test*, and the *ratio test*. The last step is to use one of these tests to show that our candidate converges. We will use the ratio test, reviewed in Appendix A. Computing directly, we find

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{t^{k+1}}{(k+1)!}}{\frac{t^k}{k!}} = \lim_{k\to\infty} \frac{t^k t}{(k+1)k!} \cdot \frac{k!}{t^k} = \lim_{k\to\infty} \frac{t}{k+1} = 0, \text{ for all } t.$$

We conclude that $y(t)$ is indeed a solution. Observe that though we have developed a series representation for our solution, we have not found a closed form solution. (What is the closed form solution?) △

Idea of the proof of Theorem 4.1, higher order equations and systems. We consider the ODE

$$\begin{aligned} y''(t) + y(t) &= 0; \\ y(0) = 0; \qquad y(1) &= 1. \end{aligned} \tag{4.2}$$

In order to proceed as above and write (4.2) as an integral equation, we first write it in the notation of Theorem 4.2 by making the substitutions $y_1(t) = y(t)$ and $y_2(t) = y'(t)$:

$$\frac{dy_1}{dt} = y_2; \quad y_1(0) = 0$$
$$\frac{dy_2}{dt} = -y_1; \quad y_2(0) = 1.$$

(Notice that the assumptions of Theorem 3.3 clearly hold for this equation.) Integrating, we obtain the integral equations,

$$y_1(t) = \int_0^t y_2(s)ds$$
$$y_2(t) = 1 - \int_0^t y_1(s)ds.$$

Our first three iterations become,

$$\begin{aligned}
y_1(t)^{(1)} &= \int_0^t 1ds = t \\
y_2(t)^{(1)} &= 1 - \int_0^t 0ds = 1 \\
y_1(t)^{(2)} &= \int_0^t 1ds = t \\
y_2(t)^{(2)} &= 1 - \int_0^t sds = 1 - \frac{t^2}{2} \\
y_1(t)^{(3)} &= \int_0^t (1 - \frac{s^2}{2})ds = t - \frac{t^3}{3!} \\
y_2(t)^{(3)} &= 1 - \int_0^t sds = 1 - \frac{t^2}{2}.
\end{aligned}$$

(By the way, I never said this was the world's most efficient algorithm.) Continuing, we find that

$$y(t) = y_1(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - ... = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{t^{2k-1}}{(2k-1)!}.$$

Again, we can apply the ratio test to determined that this series converges (to what?). □

Proof of Theorem 4.1. As with Theorem 3.1, we will only prove Theorem 4.1 in the case of single equations. The proof in the case of systems actually looks almost identical, where each statement is replaced by a vector generalization. I should mention at the outset that this is by far the most technically difficult proof of the semester. Not only is the argument itself fairly subtle, it involves a number of theorems from advanced calculus (e.g. M409).

Integrating equation (4.1), we obtain the integral equation

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s))ds.$$

Iterating exactly as in the examples above, we begin with y_0 and compute y_1, y_2,... according to

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds; \quad n = 0, 1, 2, ...$$

As both a useful calculation and a warmup for the argument to come, we will begin by estimating $\|y_1 - y_0\|$, where where $\|\cdot\|$ is defined similarly as in Theorem A.6 by

$$\|y(t)\| := \sup_{|t-t_0|<\tau} |y(t)|.$$

We compute

$$|y_1(t) - y_0| = |\int_{t_0}^{t} f(s, y_0)ds|.$$

Our theorem assumes that f is continuous on $s \in [t_0, t]$ and hence bounded, so there exists some constant M so that $\|f(s, y_0)\| \leq M$. We have, then

$$|y_1(t) - y_0| \leq \tau M.$$

Observing that the right-hand side is independent of t we can take supremum over t on both sides to obtain

$$\|y_1 - y_0\| \leq \tau M.$$

Finally, since we are at liberty to take τ as small as we like we will choose it so that $0 < \tau \leq \epsilon/M$, for some $\epsilon > 0$ to be chosen. In this way, we can insure that our new value y_1 remains in our domain of continuity of f.

We now want to look at the difference between two successive iterations and make sure the difference is getting smaller—that our iteration is actually making progress. For $|t - t_0| < \tau$, we have

$$\begin{aligned}
|y_{n+1}(t) - y_n(t)| =& |\int_{t_0}^{t} \Big(f(s, y_n(s)) - f(s, y_{n-1}(s))\Big)ds| \\
&\leq \int_{t_0}^{t} L|y_n(s) - y_{n-1}(s)|ds \leq L\tau \sup_{|t-t_0|\leq\tau} |y_n(t) - y_{n-1}(t)|.
\end{aligned}$$

Taking supremum over both sides (and observing that t has become a dummy variable on the right-hand side), we conclude

$$\|y_{n+1} - y_n\| \leq L\tau\|y_n - y_{n-1}\|,$$

Since τ is to be taken arbitrarily small, we can choose it to be as small as we like, and take $0 < \tau \leq L/2$. In this way, we have

$$\|y_{n+1} - y_n\| \leq \frac{1}{2}\|y_n - y_{n-1}\|.$$

We see that, indeed, on such a small interval of time our iterations are getting better. In fact, by carrying this argument back to our initial data, we find

$$\begin{aligned}
\|y_{n+1} - y_n\| \leq& \frac{1}{2}\|y_n - y_{n-1}\| \leq \frac{1}{2} \cdot \frac{1}{2}\|y_{n-1} - y_{n-2}\| \\
\leq& \frac{1}{2^n}\|y_1 - y_0\| \leq \frac{\epsilon}{2^n}.
\end{aligned}$$

In this way, we see that for $n > m$

$$\begin{aligned}\|y_n - y_m\| =& \| \sum_{k=m}^{n-1} (y_{k+1} - y_k) \| \leq \sum_{k=m}^{n-1} \|y_{k+1} - y_k\| \\ \leq& \epsilon \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{\epsilon}{2^{m-1}}.\end{aligned}$$

We conclude that

$$\lim_{n>m\to\infty} \|y_n - y_m\| = \lim_{n>m\to\infty} \frac{\epsilon}{2^{m-1}} = 0,$$

and thus by Cauchy's Convergence Condition (Theorem A.6) $y_n(t)$ converges to some function $y(t)$, which is our solution. $\square$

Fundamental Theorems

One of the most useful theorems from calculus is the Implict Function Theorem, which addresses the question of existence of solutions to algebraic equations. Instead of stating its most general version here, we will state exactly the case we use.

Theorem A.1. (Implicit Function Theorem) Suppose the function $f(x_1, x_2, ..., x_n)$ is C^1 in a neighborhood of the point $(p_1, p_2, ..., p_n)$ (the function is continuous at this point, and its derivatives with respect to each variable are also continuous at this point). Suppose additionally that

$$f(p_1, p_2, ..., p_n) = 0$$

and

$$\partial_{x_1} f(p_1, p_2, ..., p_n) \neq 0.$$

Then there exists a neighborhood N_p of $(p_2, p_3, ..., p_n)$ and a function $\phi : N_p \to \mathbb{R}$ so that

$$p_1 = \phi(p_2, p_3, ..., p_n),$$

and for every $x \in N_p$,

$$f(\phi(x_2, x_3, ..., x_n), x_2, x_3, ..., x_n) = 0.$$

Theorem A.2. (Mean Value Theorem) Suppose $f(x)$ is a differentiable function on the interval $x \in [a, b]$. There there exists some number $c \in [a, b]$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Theorem A.3. (Extreme Value Theorem) Suppose $f(x)$ is a function continuous on a closed interval $x \in [a, b]$. Then $f(x)$ attains a bounded absolute maximum value $f(c)$ and a bounded absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Theorem A.4. (The Ratio Test) For the series $\sum_{k=1}^{\infty} a_k$, if $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L < 1$, then the series is absolutely convergent (which means that not only does the series itself converge, but a series created by taking absolute values of the summands in the series also converges). On the other hand, if $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = L > 1$ or $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \infty$ the series diverges. if $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$, the ratio test is inconclusive.

Theorem A.5. (Cauchy's Convergence Condition) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of points and consider the limit $\lim_{n\to\infty} a_n$. A necessary and sufficient condition that this limit be convergent is that

$$\lim_{n>m\to\infty} |a_n - a_m| = 0.$$

Theorem A.6. (Cauchy's Convergence Condition for functions, in exactly the form we require) Let the series of functions $\{y_n(t)\}_{n=1}^{\infty}$ be defined for $t \in [a,b]$, and define $\|\cdot\|$ by the relation

$$\|y(t)\| := \sup_{t\in[a,b]} |y(t)|.$$

Then if

$$\lim_{n>m\to\infty} \|y_n(t) - y_m(t)\| = 0,$$

we have

$$\lim_{n\to\infty} y_n(t)$$

converges uniformly on $t \in [a,b]$.

Theorem A.7. (Taylor expansion with remainder) Suppose $f(x)$ and its first n derivatives are continuous for $x \in [a,b]$, and suppose the $(n+1)$st derivative $f^{(n+1)}(x)$ exists for $x \in (a,b)$. Then there is a value $X \in (a,b)$ so that

$$f(b) = f(a) + f'(a)(b-a) + ... + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(X)}{(n+1)!}(b-a)^{n+1}.$$

References

[BD] W. E. Boyce and R. C. Diprima, Elementary differential equations and boundary value problems, 9th edition, Wiley 2009.

[Evans] L. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics **19** (1998) American Mathematical Society.

[Hartman] P. Hartman, *A lemma in the theory of structural stability of differential equations*, Proceedings of the AMS **11** (1960) 610–620.

[H.1] www.math.tamu.edu˜/phoward/M442.html

[H.2] P. Howard, Modeling with ODE, Available at www.math.tamu.edu˜/phoward/M442.html

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