M647 Spring 2023, Practice Problems for the Final Exam

The final exam for M647 will be Monday, May 8, 10:30 a.m. - 12:30 p.m. in Blocker 160 (the usual classroom). The final will cover material from the second half of the semester: modeling with ODE, including population dynamics, chemical reactions, Newtonian mechanics, Lagrangian mechanics, and Hamiltonian mechanics; solving ODE in MATLAB, including initial value problems, event location, boundary value problems, and parameter estimation; modeling with PDE, including models in one and multiple space dimensions.

The exam will consist of two parts: Part 1 will not require MATLAB, while Part 2 will require MATLAB. Students will have to turn in Part 1 before starting Part 2, but for Part 2 students will have access to all M-files we've used this semester, from both lecture and homework. Students will be expected to access data files from the course web site.

Office hours the week before the exam will be as follows: Wednesday, May 3, 2:00 p.m. – 3:00 p.m.; Thursday, May 4, 3:30 p.m. – 4:30 p.m.; Friday, May 5, 2:00 p.m. – 3:00 p.m.

The problems below are intended to provide students with some additional practice on modeling with PDE. They are not assigned to be turned in, and solutions are included.

Practice Problems

1. Consider a fluid flowing through a cylindrical pipe with constant cross section A, velocity $v(x,t)$, density $\rho(x,t)$, specific internal energy $e(x,t)$, temperature $T(x,t)$, under pressure $p(x,t)$, and subject to viscous stress. (By *specific* internal energy, we mean internal energy per unit mass. Internal energy arises from intermolecular collisions in the fluid, and should be distinguished from the kinetic energy associated with the macroscopic fluid motion (i.e., $\frac{1}{2}mv^2$). Potential energy will not play a role in this problem.) By conserving energy, show that

$$\left[\rho(\frac{v^2}{2}+e)\right]_t + \left[\rho ve + \frac{1}{2}\rho v^3 - \kappa(x)T_x + pv - \mu v v_x\right]_x = 0, \tag{1}$$

where κ is thermal conductivity divided by A and μ is the viscosity coefficient discussed in class. This is called the Navier-Stokes energy equation.

Note. The energy density should be easy to identify. For the flux, consider each of the following, which correspond respectively with terms in (1): internal energy, kinetic energy, energy associated with heat transfer, energy associated with pressure, energy associated with viscosity.

2. Consider a mixture with two components contained in a long cylinder, and let $u(x,t)$ denote the volumetric concentration of one of the components. Assuming that mass is conserved, the concentration of the remaining component will be $1-u$. In 1958 John W. Cahn and John Hilliard suggested that under certain conditions the energy associated with such a mixture could be expressed as the functional

$$E[u] = \int_0^L F(u) + \frac{\kappa}{2}u_x^2 dx,$$

where F denotes the *bulk free energy density* of the mixture (the free energy density, assuming the entire mixture is homogeneously mixed with concentrations u and $1-u$), and $\frac{\kappa}{2}u_x^2$ is a measure of the energy associated with transitions from one concentration to another. The flux associated with u is

$$J = -M\frac{\partial}{\partial x}\frac{\delta E}{\delta u},$$

where $\frac{\delta E}{\delta u}$ is defined so that

$$E'[u]h = \int_0^L \frac{\delta E}{\delta u}h(x)dx,$$

for

$$h \in \mathcal{S}_0 := \{h \in C^2([0,L]) : h(0) = 0, \quad h(L) = 0\}.$$

Here, M is *molecular mobility*, and plays a role in this context similar to the role thermal diffusivity K plays in heat transfer. Also, $E'[u]h$ denotes variational derivative, as discussed in our section on Lagrangian mechanics. (We use J for the flux here both because it's the traditional letter in this context, and because I want to avoid confusion with the F (also traditional) we're using for bulk free energy density.) Compute $\frac{\delta E}{\delta u}$ and use it to write down a PDE for u.

3. Suppose $u(x,t)$ denotes traffic density (number of cars per unit length of road) along a certain stretch of road. In class, we discussed models in which the traffic flux depends only on traffic density u. One drawback of such models is that they do not capture a driver's reaction to what he sees ahead. For example, a driver who sees a higher density of traffic ahead will often slow down, while a driver who sees a lower density of traffic ahead will often speed up. Incorporate this idea to revise our model from class.

4. Suppose we have a dependent variable $y(t)$ that we think should approach a target function $\phi(t)$ as t increases. One simple way to model this qualitative behavior is to write down the *relaxation equation*

$$\frac{dy}{dt} = -\frac{1}{\tau}(y - \phi(t)),$$

for some contant $\tau > 0$.

a. Explain why we expect this equation to have the right qualitative behavior. Solve this equation for $y(t)$, and discuss the qualitative behavior of your solution if $\phi(t)$ is constant.

b. In Problem 3, we modeled the density u of traffic along a roadway with the equation

$$u_t + f(u)_x = \mu u_{xx}.$$

Write down an expression for the traffic velocity associated with this model in terms of u (and appropriate derivatives).

c. One criticism of the model from (b) is that it doesn't take into account the delay required for a driver to respond to surrounding road conditions. Use the idea of (a) to create a system of two equations for u and v that qualitatively accounts for this delay.

5. Suppose $\omega(t) \subset \mathbb{R}^3$ denotes an evolving region of fluid, and that $\rho(\vec{x},t)$ denotes the density of the fluid at position $\vec{x}$ and time t, $\vec{v}(\vec{x},t)$ denotes the velocity of the fluid at position $\vec{x}$ and time t, and $p(\vec{x},t)$ denotes the pressure of the fluid at position $\vec{x}$ and time t.

a. Show that

$$\vec{v} \cdot (\vec{v}^T D\vec{v}) = \vec{v} \cdot ((\vec{v} \cdot \nabla)\vec{v}).$$

Note. Here, $\vec{v}$ is regarded as a column vector, so $\vec{v}^T D\vec{v}$ denotes multiplication of a row vector by a 3×3 Jacobian matrix.

b. The kinetic energy associated with $\omega(t)$ can be computed as

$$K(t) = \frac{1}{2} \int_{\omega(t)} \rho(\vec{x}, t) |\vec{v}(\vec{x}, t)|^2 dV.$$

Show that for an inviscid fluid with no body forces

$$K'(t) = -\int_{\omega(t)} (\vec{v} \cdot \nabla p) dV.$$

Note. For Part (b), you need the following fact from class: the Navier-Stokes equations for an inviscid fluid with no body forces are

$$\begin{aligned} \rho_t + \nabla \cdot (\rho \vec{v}) &= 0 \\ \rho(\vec{v}_t + (\vec{v} \cdot \nabla)\vec{v}) &= -\nabla p. \end{aligned}$$

6. Our derivation of the Navier-Stokes momentum equation in three space dimensions was based on applying Newton's second law to a mass of fluid contained in an evolving region $\omega(t)$. The same idea can be employed in the case of one space dimension, starting with the momentum

$$p(t) = A \int_{a(t)}^{b(t)} \rho(x, t) v(x, t) dx,$$

where the interval $(a(t), b(t))$ replaces $\omega(t)$ and a fixed initial interval (a_0, b_0) replaces $\omega(0) = \omega_0$. Derive a one-dimensional version of the Reynolds Transport Theorem, and apply it to compute $\frac{dp}{dt}$ in this case.

7. Derive the wave equation for two space dimensions using Hamilton's Principle. Use a square membrane $[0, L] \times [0, M]$, and assume the potential energy is proportional to the deformation from equilibrium, measured by membrane area:

$$P = k\Big(\int_0^L \int_0^M \sqrt{1 + u_x^2 + u_y^2} dx dy - LM \Big).$$

Ignore the effect of gravity.

8. In this problem, we'll consider a single phenomemon from two different inertial reference frames, and compare the results. In order to make this comparison, we'll need to work with a general form of *electromotive force* $\mathcal{E}$, which we define to be

$$\mathcal{E} = \oint_{\mathcal{C}} \vec{f} \cdot \hat{t} dl,$$

where the integral is over a closed circuit $\mathcal{C}$ and $\vec{f}$ is the force per unit charge, which according to the Lorentz force law is

$$\vec{f} = \vec{E} + \vec{v} \times \vec{B}.$$

a. Suppose a rectangular circuit of wire is moving to the right with velocity v as depicted in Figure 1, and that it enters a constant magnetic field $\vec{B}$ directed out of the page in the interior of the dashed box. Compute the electromotive force $\mathcal{E}$ on electrons in the wire once the leading end of the wire has entered the magnetic field, and before the leading end exits the magnetic field.

b. Suppose that the situation in Part (a) is reversed so that the rectangular circuit of wire is stationary and the magnetic field is moving to the left with velocity $-v$ (so, in particular, the electrons now have zero velocity[1]). Again, compute $\mathcal{E}$.

c. Explain how the calculations in Parts (a) and (b) differ?

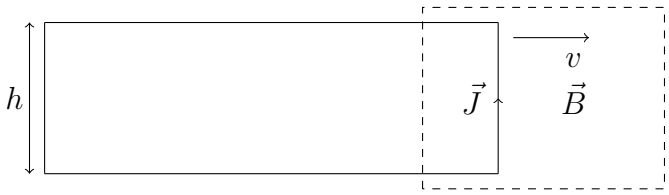

Figure 1: Wire entering a magnetic field.

Solutions

1. First, the quantity

$$u := \rho A(\frac{v^2}{2} + e)$$

is clearly (one-dimensional) energy density. We can derive (1) by identifying the energy flux. Here, A will divide out in the end.

Internal and kinetic energy. For the internal and kinetic energies, the flux is clearly vu (just as $v\rho$ is mass flux), giving the first two terms. I.e.,

$$vu = (\rho ve + \frac{1}{2}\rho v^3)A.$$

Energy associated with heat transfer. The term $-A\kappa(x)T_x$ is just Fourier's law, as discussed in class.

Energy associated with pressure. First, the easiest way to think about this is to recall that the one-dimensional energy flux is energy per unit time, and that this energy can be computed

[1]Ignoring orbital velocity.

as work done by pressure. I.e., by writing work as force times distance, and noting that force due to pressure is pA, we can identify the one-dimensional flux as

$$W \cdot \Delta t^{-1} = (pA\Delta x)\Delta t^{-1} = pA\frac{\Delta x}{\Delta t} = (pv)A.$$

Alternatively, we can check how this term would arise in a full derivation, as we did in class for the continuity equation. For this, we recall that during a time interval $[t, t+\Delta t]$, the fluid will travel a distance $v(x,t)\Delta t$, so that the work done by pressure forces is

$$W = -\Big(p(x+\Delta x,t)Av(x+\Delta x,t)\Delta t - p(x,t)Av(x,t)\Delta t\Big) \cong -(pv)_x A\Delta x\Delta t.$$

In the full derivation, the quantity $A\Delta x\Delta t$ is divided out, giving the term $-(pv)_x$ (already differentiated in this case). The term $-(pv)_x$ appears on the right-hand side of the equation in this derivation, leading to the positive term in (1).

Energy associated with viscosity. Again, the easy way first. As with pressure,

$$W \cdot \Delta t^{-1} = \mu A v_x \Delta x \Delta t^{-1} = \mu A v v_x,$$

where in this case v_x arises from $\frac{v}{h}$ in our usual definition of viscosity. Alternatively, we can proceed by writing the work done by viscous forces at the right as

$$\begin{aligned} W_r &\cong \mu\frac{A(v(x+\Delta x+h,t)-v(x+\Delta x,t))}{h}v\Delta t \\ &\cong \mu A v(x+\Delta x,t)v_x(x+\Delta x,t)\Delta t, \end{aligned}$$

and likewise the work done by viscous forces at the left by

$$\begin{aligned} W_l &\cong -\mu\frac{A(v(x-h,t)-v(x,t))}{h}v\Delta t \\ &\cong -\mu A v(x,t)v_x(x,t)\Delta t. \end{aligned}$$

Then

$$\begin{aligned} W_r + W_l &= \mu\Big(v(x+\Delta x,t)v_x(x+\Delta x,t) - v(x,t)v_x(x,t)\Big)A\Delta t \\ &\cong \mu(vv_x)_x A\Delta x\Delta t. \end{aligned}$$

Again, $A\Delta x\Delta t$ is divided out in the full derivation, and this is on the right-hand side, leading to the negative sign in (1).

2. The first step is to compute $\frac{\delta E}{\delta u}$. We proceed by writing

$$\begin{aligned} \phi(\tau) &= E[u+\tau h] \\ &= \int_0^L F(u+\tau h) + \frac{\kappa}{2}(u_x+\tau h_x)^2 - F(u) - u_x^2 dx, \end{aligned}$$

and computing

$$\phi'(\tau) = \int_0^L F'(u+\tau h)h + \kappa(u_x+\tau h_x)h_x dx.$$

We have, then,

$$E'[u]h = \phi'(0) = \int_0^L F'(u)h + \kappa u_x h_x dx.$$

For the second summand, we integrate by parts to write

$$E'[u]h = \int_0^L (F'(u) - \kappa u_{xx})h dx.$$

We conclude

$$\frac{\delta E}{\delta u} = -\kappa u_{xx} + F'(u),$$

and consequently the equation is

$$u_t - (M(-\kappa u_{xx} + F'(u))_x)_x = 0.$$

3. First, our general traffic flow model from class had the form

$$u_t + f(u)_x = 0,$$

and the specific model we considered was the Gompertz-Greenberg model with

$$f = -cu \ln(\frac{u}{u_{\max}}).$$

Now, if traffic has higher density ahead of a driver then $u_x > 0$, while if traffic has a lower density ahead of a driver then $u_x < 0$. (We are thinking of these derivatives as evaluated at the driver's current position.) If drivers who see high density ahead begin to slow down the effect will be a shift of density to the left, while if drivers who see lower density ahead begin to speed up the effect will be a shift of density to the right. That is, we replace f with a revised flux

$$\mathcal{F} = f(u) - ku_x.$$

Our general model becomes

$$u_t + f(u)_x = ku_{xx},$$

and the Gompertz-Greenberg model becomes

$$u_t - c\Big(u \ln(\frac{u}{u_{\max}})\Big)_x = ku_{xx}.$$

4. For (a), we see that if $y(t) > \phi(t)$ then y will decrease, while if $y(t) < \phi(t)$ then y will increase. This will have the effect of moving $y(t)$ toward $\phi(t)$. The equation is easily solved with an integrating factor, and we obtain

$$y(t) = e^{-t/\tau} y_0 + \frac{1}{\tau} \int_0^t e^{-(t-s)/\tau} \phi(s) ds.$$

If $\phi(t) = \phi_0$ is constant, we get

$$y(t) = e^{-t/\tau} + \phi_0(1 - e^{-t/\tau}),$$

so that $y(t) \to \phi_0$ as $t \to \infty$, as expected.

For (b), the flux is generally vu, so we need

$$f(u) - \mu u_x = vu \implies v = \frac{f(u)}{u} - \frac{\mu}{u} u_x.$$

For (c), we can allow drivers to transition toward the target velocity $v = \frac{f(u)}{u} - \frac{\mu}{u} u_x$ with a relaxation model

$$\frac{d}{dt} v(x(t), t) = -\frac{1}{\tau}(v - \frac{f(u)}{u} + \frac{\mu}{u} u_x),$$

which becomes

$$v_t + v v_x = -\frac{1}{\tau}(v - \frac{f(u)}{u} + \frac{\mu}{u} u_x).$$

We couple this with

$$\rho_t + (v\rho)_x = 0.$$

This model is discussed on pp. 72-73 of our reference by Whitham.

5. For (a),

$$(\vec{v}^T D\vec{v})_i = \sum_{j=1}^{3} v_j \frac{\partial v_j}{\partial x_i} \implies \vec{v} \cdot (\vec{v}^T D\vec{v}) = \sum_{i,j=1}^{3} v_i v_j \frac{\partial v_j}{\partial x_i}.$$

Likewise,

$$((\vec{v} \cdot \nabla)\vec{v})_i = \sum_{j=1}^{3} v_j \frac{\partial v_i}{\partial x_j} \implies \vec{v} \cdot ((\vec{v} \cdot \nabla)\vec{v}) = \sum_{i,j=1}^{3} v_i v_j \frac{\partial v_i}{\partial x_j}.$$

By switching the roles of i and j in one of these we see that they agree.

For (b), we proceed precisely as in the differentiation of $p_i(t)$ from class, with (ρv_i) replaced by $\rho|\vec{v}|^2$ to get

$$\begin{aligned} K'(t) &= \frac{1}{2} \int_{\omega(t)} \rho((|\vec{v}|^2)_t + (D|\vec{v}|^2)\vec{v}) dV \\ &= \frac{1}{2} \int_{\omega(t)} 2\vec{v} \cdot (\rho \vec{v}_t) + 2\rho\vec{v} \cdot (\vec{v}^T D\vec{v}) dV. \end{aligned}$$

We can now use the momentum equation to write

$$\rho\vec{v}_t = -\rho(\vec{v} \cdot \nabla)\vec{v} - \nabla p,$$

so that we have

$$\begin{aligned} K'(t) &= \int_{\omega(t)} \vec{v} \cdot (\rho\vec{v}_t) + \rho\vec{v} \cdot (\vec{v}^T D\vec{v}) dV \\ &= \int_{\omega(t)} \vec{v} \cdot (-\rho(\vec{v} \cdot \nabla)\vec{v} - \nabla p) + \rho\vec{v} \cdot (\vec{v}^T D\vec{v}) dV \\ &= \int_{\omega(t)} \vec{v} \cdot (-\nabla p) + \rho(-\vec{v} \cdot ((\vec{v} \cdot \nabla)\vec{v}) + \vec{v} \cdot (\vec{v}^T D\vec{v})) dV. \end{aligned}$$

Using (a), we see that the second and third summands cancel, giving the claim of (b).

6. We start off by deriving a one-dimensional Reynolds Transport Theorem, which will give us a relation for the derivative

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx,$$

where $f \in C^1(\mathbb{R}\times\mathbb{R}_+,\mathbb{R})$. Using the change of variables $x=\phi(X,t)$, this integral becomes

$$\int_{a_0}^{b_0} f(\phi(X,t),t)\phi_X(X,t)dX.$$

Similarly as we did in class, we can write

$$\begin{aligned} R(X,t) &= f(\phi(X,t),t)\\ J(X,t) &= \phi_X(X,t). \end{aligned}$$

Notice that $\phi_X > 0$, because $\phi(X,0)=X$ so that $\phi_X = 1$, and ϕ can't change sign. In this way, we have

$$\int_{a_0}^{b_0} R(X,t)J(X,t)dX,$$

and we easily compute

$$\frac{d}{dt}\int_{a_0}^{b_0} R(X,t)J(X,t)dX = \int_{a_0}^{b_0} R_t(X,t)J(X,t)+R(X,t)J_t(X,t)dX.$$

For $R(X,t)$ we compute

$$\begin{aligned} R_t(X,t) &= \frac{d}{dt}f(\phi(X,t),t) = f_x(\phi(X,t),t)\frac{\partial\phi}{\partial t}(X,t)+f_t(\phi(X,t),t)\\ &= f_t(\phi(X,t),t)+f_x(\phi(X,t),t)v(\phi(X,t),t). \end{aligned}$$

Likewise for $J(X,t)=\phi_X(X,t)$,

$$\begin{aligned} J_t(X,t) &= \frac{\partial}{\partial t}\phi_X(X,t) = \frac{\partial}{\partial X}\phi_t(X,t) = \frac{\partial}{\partial X}v(\phi(X,t),t)\\ &= v_x(\phi(X,t),t)\phi_X(X,t) = v_x(\phi(X,t),t)J(X,t). \end{aligned}$$

We see that

$$\begin{aligned} \frac{d}{dt}\int_{a_0}^{b_0} R(X,t)J(X,t)dX &= \int_{a_0}^{b_0}\Big\{f_t(\phi(X,t),t)+f_x(\phi(X,t),t)v(\phi(X,t),t)\\ &\quad + v_x(\phi(X,t),t)f(\phi(X,t),t)\Big\}\phi_X(X,t)dX\\ &= \int_{a(t)}^{b(t)} f_t(x,t)+(vf)_x(x,t)dx. \end{aligned}$$

This is the one-dimensional Reynolds Transport Theorem.

We now just apply this to the momentum

$$p(t) = A\int_{a(t)}^{b(t)} \rho(x,t)v(x,t)dx,$$

to see that

$$\begin{aligned}\frac{dp}{dt} &= A\int_{a(t)}^{b(t)} (\rho v)_t + (\rho v^2)_x dx \\ &= A\int_{a(t)}^{b(t)} \rho_t v + \rho v_t + v(\rho v)_x + \rho v v_x dx \\ &= A\int_{a(t)}^{b(t)} \rho(v_t + v v_x)dx\end{aligned}$$

7. The kinetic energy is

$$K = \int_0^L \int_0^M \frac{1}{2}\rho(x,y)u_t^2 dxdy,$$

so the Lagrangian is

$$L = \int_0^L \int_0^M \frac{1}{2}\rho(x,y)u_t^2 - k\Big(\sqrt{1+u_x^2+u_y^2} - 1\Big)dxdy.$$

The action is

$$A[u] = \int_0^T \int_0^L \int_0^M \frac{1}{2}\rho(x,y)u_t^2 - k\Big(\sqrt{1+u_x^2+u_y^2} - 1\Big)dxdydt.$$

If we denote our domain of u

$$\Omega = [0,L]\times[0,M]\times[0,T],$$

then it's reasonable to take the domain of A to be the function space

$$\mathcal{S} := \{u \in C^2(\Omega) : u|_{\partial\Omega} = u_b = \text{specified}\}.$$

According to Hamilton's Principle, we should have

$$A'[u] = 0.$$

For any h in the function space

$$\mathcal{S}_0 = \{u \in C^2(\Omega) : u|_{\partial\Omega} = 0\}.$$

We set

$$\begin{aligned}\phi(\tau) &= A[u+\tau h] \\ &= \int_0^T \int_0^L \int_0^M \frac{1}{2}\rho(x,y)(u_t+\tau h_t)^2 - k\Big(\sqrt{1+(u_x+\tau h_x)^2+(u_y+\tau h_y)^2} - 1\Big)dxdydt,\end{aligned}$$

and compute

$$\phi'(\tau) = \int_0^T \int_0^L \int_0^M \rho(x,y)(u_t + \tau h_t)h_t - k\frac{(u_x + \tau h_x)h_x + (u_y + \tau h_y)h_y}{\sqrt{1 + (u_x + \tau h_x)^2 + (u_y + \tau h_y)^2}} dxdydt$$

We see that

$$A'[u]h = \phi'(0) = \int_0^T \int_0^L \int_0^M \rho(x,y)u_t h_t - k\frac{u_x h_x + u_y h_y}{\sqrt{1 + u_x^2 + u_y^2}} dxdydt.$$

Upon integrating the first integrand by parts in t, the second by parts in x, and the third by parts in y, we find

$$\int_0^T \int_0^L \int_0^M \Big[-\rho(x,y)u_{tt} + k\Big(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\Big)_x + k\Big(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\Big)_y \Big] h dxdydt = 0.$$

As usual, our freedom to choose h ensures that its multiplier in the integrand must be 0. I.e.,

$$-\rho(x,y)u_{tt} + k\Big(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\Big)_x + k\Big(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\Big)_y = 0,$$

which becomes the wave equation when we assume u_x and u_y are both small and take $\rho(x,y)$ constant.

8. For (a), once the right side of the wire has entered the magnetic field, electrons in the wire are affected by the Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$, where q denotes the charge on an electron. If we take the x-direction to be out of the page and the y-direction to be the direction of motion, then $\vec{v} = (0, v, 0)$ and $\vec{B} = (|\vec{B}|, 0, 0)$, so that

$$\vec{v} \times \vec{B} = -v|\vec{B}|\hat{k}.$$

I.e., $\vec{f} = -v|\vec{B}|\hat{k}$, so

$$\mathcal{E} = \oint_{\mathcal{C}} \vec{f} \cdot \hat{t} dl = \mathcal{E} = \oint_{\mathcal{C}} (-v|\vec{B}|)\hat{k} \cdot \hat{t} dl = -v|\vec{B}|h,$$

where the sole contribution has been from the front edge of the wire.

For (b), let's suppose for convenience that the magnetic field arrives at the right edge of the wire at time $t = 0$. Subsequently, the magnetic flux through the wire will be

$$\int_\Sigma \vec{B} \cdot \hat{n} dS = |\vec{B}|hvt,$$

where hvt is the area of the wire that has entered the magnetic field. According to Faraday's Law, this means

$$\int_{\partial\Sigma} \vec{E} \cdot \hat{t} dl = -\frac{d}{dt}(|\vec{B}|hvt) = -|\vec{B}|hv.$$

Summing up with (c), we see that $\mathcal{E}$ is the same in both cases, though seemingly for different reasons. For (a), there are moving charges with no electric field, while for (b) the charge isn't moving, but there is an electric field. Since these arise simply as different points of view on the same problem, it seems natural to wonder if electricity and magnetism are simply two manifestations of the same phenomenon.[2]

[2]They are.