M647, Spring 2023, Practice Problems for the Midterm

The midterm exam for M647 will be Thursday March 9, 7:00-9:00 p.m. in Blocker 628. The exam will cover the following topics: linear and nonlinear regression, including theory and implementation; dimensional analysis, including theory and implementation (both with regression and structured experiments); and non-dimensionalization.

The exam will consist of two parts: Part 1 will not require MATLAB, while Part 2 will require MATLAB. Students will not be allowed to use notes or references for Part 1, and students must submit Part 1 before beginning Part 2. For Part 2, students can access any notes, references, M-files etc., except no communication between students is allowed. Part 2 should be submitted via Canvas.

For some questions, students may be expected to access data files from the course web site.

The problems in homework assignments 2 through 6 serve as good practice for the exam, and solutions can be found in Canvas under the *files* link. The problems below are intended to provide students with some additional practice. These problems are not assigned to be turned in, and solutions are included.

Practice Problems

1. Suppose data $\{(x_k, y_k)\}_{k=1}^N$ is fit by linear least squares regression to a line through the origin $y = mx$. Determine whether or not the relation $\mu_y = m\mu_x$ must hold in all such cases. Here μ_x and μ_y are respectively means of the values $\{x_k\}_{k=1}^N$ and $\{y_k\}_{k=1}^N$.

2. Suppose F is a design matrix with linearly independent columns, and set $V = (F^T F)^{-1}$.

a. Show that $V = (VF^T)(VF^T)^T$.

b. Show that the diagonal entries of V are all positive.

3. When an object such as a marble falls through a viscous fluid such as oil its terminal velocity depends on the marble's radius r, the gravitational acceleration $g = 9.81\ m/s^{-2}$, the fluid viscosity μ, and the density difference $\Delta\rho = \rho_m - \rho_f$, where ρ_m denotes marble density and ρ_f denotes fluid density.

a. Find a general relationship for the dependence of v on the variables r, g, μ, and $\Delta\rho$.

b. Use the data $\{(r_k, g_k, \mu_k, d\rho_k, v_k)\}_{k=1}^{10}$ in the MATLAB M-file *marbledata.m* (available on the course web site) to complete your model from Part (a).

c. Use your model to predict v for the case $r = .0065$, $\mu = 15 kg\cdot m^{-1}s^{-1}$, $\Delta\rho = 672.8 kg\cdot m^{-3}$. This experiment was carried out during the undergraduate modeling class Fall 2009 (Dial soap, marble, tennis ball can), and we timed $v = .0097 m/s$.

4. Suppose the power P generated by a windmill depends on the following five variables (and no others): density of air ρ, viscosity of air μ, radius of the windmill r, wind speed v, and fraction of kinetic energy transferred into electrical energy available to the grid η. (Here, $[P] = ML^2T^{-3}$, $[\rho] = ML^{-3}$, $[\mu] = ML^{-1}T^{-1}$, $[r] = L$, $[v] = LT^{-1}$, and $[\eta] = 1$).

a. Find a general form for the dependence of P on the other variables.

b. Suppose you would like to compute the power P for a windmill with $\rho = 1.25$ kg/m^3, $\mu = 2.00 \times 10^{-5}$ kg/(m$\cdot$s), $r = 100$ m, $v = 10$ m/s, and $\eta = .5$, but you would like to carry out your experiment with a model windmill with radius $r = 2m$. Assuming you can design your experiment with any values you like for the other variables (i.e., ρ, μ, v, and η), design an experiment that will allow you to compute P.

5. Solve the following:

a. Before proving Buckingham's Theorem, we worked through the steps of the proof with our example in which an object is fired directly up from a height h. During that calculation, we noted that the exponent vectors corresponding with $\pi_1 = \frac{gh}{v^2}$ and $\pi_2 = \frac{gt}{v}$ are

$$\vec{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}; \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Starting with $\vec{v}_1$ and $\vec{v}_2$, we completed a basis for $\mathbb{R}^4$ by adding the vectors

$$\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad \vec{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

which correspond with dimensioned products $\pi_3 = h$ and $\pi_4 = t$. We stated that it must be possible to write each of our four variables v, g, h, t in terms of $\{\pi_j\}_{j=1}^4$. Write out what these combinations would be.

b. Write out the equation

$$-\frac{1}{2}\frac{gt^2}{h} + \frac{vt}{h} + 1 = 0$$

by using Part (a) to replace each of the variables v, g, h, t with its appropriate combination of $\{\pi_j\}_{j=1}^4$. What happens to π_3 and π_4? How did you know this had to happen?

c. Alternatively (to the approach taken in class), we could have completed a basis for $\mathbb{R}^4$ by adding the vectors

$$\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad \vec{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

Explain how our general proof would have accomodated this choice. Explicitly compute the updated choices of π_3 and π_4 we would get in the proof.

6. Consider an object of mass m moving along a frictionless surface and attached to a wall by a spring. If we let $y(t)$ denote the spring's displacement from equilibrium, and we assume $y(t)$ is small, then the motion can often be modeled by an equation of the form

$$my'' = -k_1 y + k_3 y^3,$$

where k_1 is Hooke's constant and the term k_3y^3 is a nonlinear correction corresponding with the fact that the spring will weaken if either strongly stretched or strongly compressed. Find the dimensions of k_1 and k_3 and nondimensionalize this equation.

7. Non-dimensionalize the system

$$\begin{aligned}\frac{d^2x}{dt^2} &= -b\sqrt{x'^2+y'^2}x' \\ \frac{d^2y}{dt^2} &= -g-b\sqrt{x'^2+y'^2}y',\end{aligned}$$

where x and y denote coordinates of a position vector, g denotes the usual gravitational constant, and b denotes a drag coefficient. Write down initial conditions for your non-dimensional system, based on the original initial conditions,

$$\begin{aligned}x(0) &= 0 \\ y(0) &= h_0 \\ x'(0) &= |\vec{v}_0|\cos\theta \\ y'(0) &= |\vec{v}_0|\sin\theta.\end{aligned}$$

Solutions

1. First, for the regression calculation the design matrix will be

$$F = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}.$$

The regression slope m will be a solution of the normal equation $F^TFm = F^T\vec{y}$, which in this case becomes

$$(\sum_{k=1}^{N} x_k^2)m = \sum_{k=1}^{N} x_k y_k.$$

This is not the relation $\mu_y = m\mu_x$, but to make things concrete we should verify that we can find an example for which $\mu_y \neq m\mu_x$. For this, let's take two points $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (1, 2)$. Then

$$\sum_{k=1}^{N} x_k^2 = 1, \quad \text{and} \quad \sum_{k=1}^{N} x_k y_k = 2,$$

so $m = 2$. On the other hand, $\mu_x = 1/2$ and $\mu_y = 3/2$, so $\mu_y = 3\mu_x$. In particular, $\mu_y \neq 2\mu_x$, so we see that the relation doesn't generally hold.

2. For (a), we compute

$$(VF^T)(VF^T)^T = VF^TFV^T = V(F^TF)(F^TF)^{-1} = V.$$

For (b), we observe that for any matrix A, the diagonal entries of the product AA^T are always the Euclidean norms of the row vectors of A, and these norms must be positive unless there is a row of zeros. For $A = VF^T$, we get positive values unless $(F^TF)^{-1}F^T$ has a row of zeros. This would mean the matrix $F(F^TF)^{-1}$ has a column of zeros, which cannot be true because $\vec{v} = 0$ is the only vector in $\mathcal{N}(F(F^TF)^{-1})$. (This is because $\mathcal{N}(F) = 0$, by linear independence of its columns, and so $\vec{v} \in \mathcal{N}(F(F^TF)^{-1})$ if and only if $(F^TF)^{-1}\vec{v} = 0$, which implies $\vec{v} = 0$.)

3. We begin by looking for dimensionless products

$$\pi = \pi(r, g, \mu, \Delta\rho, v) = r^a g^b \mu^c \Delta\rho^d v^e,$$

with dimensions

$$1 = L^a L^b T^{-2b} M^c L^{-c} T^{-c} M^d L^{-3d} L^e T^{-e},$$

and dimensions equations

$$\begin{aligned} L &: 0 = a + b - c - 3d + e \\ T &: 0 = -2b - c - e \\ M &: 0 = c + d. \end{aligned}$$

In matrix form this is

$$\begin{pmatrix} 1 & 1 & -1 & -3 & 1 \\ 0 & -2 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In row reduced echelon form,

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We have

$$\begin{aligned} a &= \frac{3}{2}d - \frac{1}{2}e \\ b &= \frac{1}{2}d - \frac{1}{2}e \\ c &= -d. \end{aligned}$$

For π_1, we choose $e = 0$ and $d = 1$, which gives $a = \frac{3}{2}$, $b = \frac{1}{2}$, and $c = -1$. We conclude

$$\pi_1 = \frac{r^{3/2} g^{1/2} \Delta\rho}{\mu}.$$

For π_2, we choose $e = 1$ and $d = 0$, which gives $a = -\frac{1}{2}$, $b = -\frac{1}{2}$, and $c = 0$. We conclude

$$\pi_2 = \frac{v}{\sqrt{rg}}.$$

Buckingham's Theorem and the Implicit Function Theorem suggest there exists a function ϕ so that

$$\pi_2 = \phi(\pi_1),$$

or equivalently

$$\frac{v}{\sqrt{rg}} = \phi(\frac{r^{3/2}g^{1/2}\Delta\rho}{\mu}) \Rightarrow v = \sqrt{rg}\phi(\frac{r^{3/2}g^{1/2}\Delta\rho}{\mu}).$$

The fit for Part (b) is carried out in the MATLAB M-file *marblefit.m*.

```
%MARBLEFIT: MATLAB script M-file that fits the data in
%marbledata to a choice of dimensionless products
%
%define data
marbledata;
pi1 = delrho.*r.^(3/2)*sqrt(g)./mu;
pi2 = v./sqrt(r*g);
p = polyfit(pi1,pi2,1)
plot(pi1,pi2,'o',pi1,p(1)*pi1+p(2))
axis equal
title('Plot of \pi_2 vs. \pi_1 for Marbles Data','FontSize',15)
%
%model
v = @(r,g,mu,delrho) p(1)*r^2*g*delrho/mu+p(2)*sqrt(r*g);
%prediction
v(.0065,9.81,15,672.8)
```

The plot this creates appears in Figure 1.

We find that

$$\phi(\pi_1) = .1283\pi_1 + .0317,$$

and so

$$v = .1283(\frac{r^2 g\Delta\rho}{\mu}) + .0317\sqrt{rg}.$$

For Part (c), we find $v = .0104$.

By the way, it's easy to verify, using Newtonian mechanics, that the theoretical expression for v is

$$v = \frac{2r^2 g\Delta\rho}{9\mu}.$$

This would correspond with $p_1 = \frac{2}{9}$ and $p_2 = 0$ in our fit.

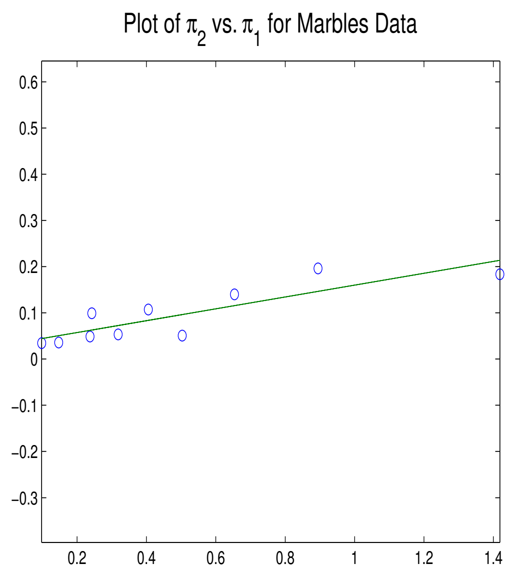

Figure 1: Figure for the marbles data.

4. First, we look for dimensionless products

$$\pi = \pi(\rho, \mu, r, v, \eta, P) = \rho^a \mu^b r^c v^d \eta^e P^f,$$

which gives

$$1 = M^a L^{-3a} M^b L^{-b} T^{-b} L^c L^d T^{-d} M^f L^{2f} T^{-3f}.$$

This gives the dimensions equations

$$\begin{aligned} L &: 0 = -3a - b + c + d + 2f \\ M &: 0 = a + b + f \\ T &: 0 = -b - d - 3f. \end{aligned}$$

This corresponds with the matrix equation

$$\begin{pmatrix} -3 & -1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Using MATLAB to put this matrix in row reduced echelon form we find

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix}.$$

This gives

$$\begin{aligned} a &= d + 2f \\ b &= -d - 3f \\ c &= d + f. \end{aligned}$$

For π_1 we take $e = 1$ and all the other exponents 0, so that $\pi_1 = \eta$. For π_2 we take $f = 0$, $e = 0$, and $d = 1$ to obtain $\pi_2 = \frac{\rho v r}{\mu}$ (the Reynold's number). For π_3 we take $f = 1$, $e = 1$, and $d = 1$ to obtain $\pi_3 = \frac{rP\rho^2}{\mu^3}$. By Buckingham's Theorem and the Implicit Function Theorem we expect there to exist a function ϕ so that

$$\pi_3 = \phi(\pi_1, \pi_2),$$

or

$$P = \frac{\mu^3}{r\rho^2}\phi(\eta, \frac{\rho v r}{\mu}).$$

For Part (b) we want to choose $r_e = 2$ and values ρ_e, μ_e, v_e, and η_e, so that

$$\begin{aligned} \eta_e &= \eta = .5 \\ \frac{\rho_e v_e r_e}{\mu_e} &= \frac{\rho v r}{\mu} = 6.25 \times 10^7. \end{aligned}$$

Since $r_e = 2$ is fixed, we must choose ρ_e, v_e, and μ_e so that

$$\frac{\rho_e v_e}{\mu_e} = 3.1250 \times 10^7.$$

If we want to stick with the same fluid (air), we can set

$$v_e = \frac{3.1250 \times 10^7 \times 2.00 \times 10^{-5}}{1.25} = 500 m/s.$$

(This large value suggests that we might want to work with a different fluid, but that's not necessary for the problem.)

5. For (a) the most systematic thing to do is to notice that the system $\vec{v} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4$ can be expressed as

$$\vec{v} = M\vec{\alpha},$$

where $M = (\vec{v}_1\, \vec{v}_2\, \vec{v}_3\, \vec{v}_4)$. For v we need $\vec{v} = (1, 0, 0, 0)^T$, and we find $\vec{\alpha} = (-1, 1, 1, -1)$, giving

$$v = \frac{\pi_2\pi_3}{\pi_1\pi_4}.$$

Likewise, for g we need $\vec{v} = (0, 1, 0, 0)^T$, and we find $\vec{\alpha} = (-1, 2, 1, -2)$, giving

$$g = \frac{\pi_2^2\pi_3}{\pi_1\pi_4^2}.$$

The last two are immediate, $h = \pi_3$ and $t = \pi_4$.

For (b), we write out

$$-\frac{1}{2}\frac{\pi_2^2\pi_3}{\pi_1\pi_4^2}\cdot\frac{\pi_4^2}{\pi_3}+\frac{\pi_2\pi_3}{\pi_1\pi_4}\cdot\frac{\pi_4}{\pi_3}+1=-\frac{\pi_2^2}{\pi_1}+\frac{\pi_2}{\pi_1}+1.$$

We see that π_3 and π_4 cancel out, and we knew this had to happen, because ψ is independent of π_3 and π_4.

For (c), the dimensioned products associated with $\vec{v}_3$ and $\vec{v}_4$ would be

$$\pi_3 = vght; \quad \pi_4 = \frac{vht}{g}.$$

We see that $[\pi_3] = L^3T^{-2}$ and $[\pi_4] = LT^2$. Proceeding as in our proof, we associate with π_3 and π_4 the vectors $\vec{u}_3 = (3, -2)$ and $\vec{u}_4 = (1, 2)$. We set

$$B = \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$$

and reduce to row echelon form

$$\tilde{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(We're using B here, because A has a specific meaning for us.) This now corresponds with a new pair of dimensioned products, the first with dimension length and the second with dimension time.

Explicitly, the first step of the row reduction would be to divide the first row by 3, and this corresponds with replacing π_3 with $\pi_3^{(1)} = (vght)^{1/3}$. For the second step, we would multiple (the new) row 1 by -1 and add the result to row 2. This corresponds with replacing π_4 with $\pi_4^{(1)} = \pi_4/\pi_3 = v^{2/3}g^{-4/3}h^{2/3}t^{2/3}$. At this point, we have

$$\hat{B} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{8}{3} \end{pmatrix}.$$

We multiply the second row by $\frac{3}{8}$, which corresponds with replacing $\pi_4^{(1)}$ with $\pi_4^{(2)} = v^{1/4}g^{-1/2}h^{1/4}t^{1/4}$. And last we multiply row 2 by $\frac{2}{3}$ and add the result to the first row, which corresponds with replacing $\pi_3^{(1)}$ with $\pi_3^{(2)} = \pi_3^{(1)}(\pi_4^{(2)})^{2/3} = (vht)^{1/2}$. I.e., we get to

$$\begin{aligned} \pi_3^{(2)} &= (vht)^{1/2}; \quad [\pi_3^{(2)}] = L \\ \pi_4^{(2)} &= v^{1/4}g^{-1/2}h^{1/4}t^{1/4}; \quad [\pi_4^{(2)}] = T. \end{aligned}$$

Additional notes. For Part (c), the goal above was to mimic our proof from class of Buckingham's Theorem, but it might also be instructive to see how this plays out with linear combinations of vectors rather than row operations. (Keeping in mind that the two things are ultimately equivalent.) First, for $\vec{u}_3$ and $\vec{u}_4$ above, we emphasize the correspondence

$$\begin{aligned} \vec{u} &= \alpha_1\vec{u}_3 + \alpha_2\vec{u}_4 \\ \pi &= \pi_3^{\alpha_1}\pi_4^{\alpha_2}. \end{aligned}$$

I.e., this works just as with vectors $\vec{v}$ associated with the variables in dimensionless products. In reducing B to $\tilde{B}$, the first row operation is to replace the first row in B with the row divided by 1/3. In terms of $\vec{u}_3$ and $\vec{u}_4$, this corresponds with replacing $\vec{u}_3$ with $\vec{u}_3^{(1)} = \frac{1}{3}\vec{u}_3 = (1, -2/3)$, and in terms of π_3 and π_4 this corresponds with replacing π_3 with $\pi_3^{(1)} = \pi_3^{1/3} = (vght)^{1/3}$. The next row operation is to multiply the first row by -1 and add the result to the second row. In this case, we are replacing $\vec{u}_4$ with $\vec{u}_4^{(1)} = -\vec{u}_3^{(1)} + \vec{u}_4 = (0, 8/3)$. Correspondingly, we replace π_4 with $\pi_4^{(1)} = (\pi_3^{(1)})^{-1}\pi_4 = (vght)^{-1/3}(vht/g) = (vht)^{2/3}/g^{4/3}$. Notice that after this step

$$
\begin{aligned}
[\pi_3^{(1)}] &= [(vght)^{1/3}] = LT^{-2/3} \\
[\pi_4^{(1)}] &= [(vht)^{2/3}/g^{4/3}] = T^{8/3},
\end{aligned}
$$

as expected. The remaining row operations can be explained similarly.

6. First, $[k_1 y] = MLT^{-2}$, so $[k_1] = MT^{-2}$. Likewise, $[k_3 y^3] = MLT^{-2}$, so $[k_3] = ML^{-2}T^{-2}$. Now we set

$$\tau = \frac{t}{A}, \quad Y(\tau) = \frac{y(t)}{B},$$

so that

$$y'(t) = B\frac{d}{dt}Y(\tau) = BY'(\tau)\frac{d\tau}{dt} = \frac{B}{A}Y'(\tau),$$

and

$$y''(t) = \frac{B}{A^2}Y''(\tau).$$

We have, then,

$$m\frac{B}{A^2}Y'' = -k_1 BY + k_2 B^3 Y^3 \Rightarrow Y'' = -k_1\frac{A^2}{m}Y + k_2\frac{A^2B^2}{m}Y^3.$$

If we choose $A = \sqrt{\frac{m}{k_1}}$ and $B = \sqrt{\frac{k_1}{k_2}}$ (with dimensions $[A] = T$ and $[B] = L$) we have

$$Y'' = -Y + Y^3.$$

7. We introduce the non-dimensional variables

$$\tau = \frac{t}{A}; \quad X(\tau) = \frac{x(t)}{B}; \quad Y(\tau) = \frac{y(t)}{C},$$

so that the equations become

$$
\begin{aligned}
\frac{B}{A^2}\frac{d^2X}{d\tau^2} &= -b\sqrt{\frac{B^2}{A^2}X'^2 + \frac{C^2}{A^2}Y'^2}\,\frac{B}{A}X' \\
\frac{C}{A^2}\frac{d^2Y}{d\tau^2} &= -g - b\sqrt{\frac{B^2}{A^2}X'^2 + \frac{C^2}{A^2}Y'^2}\,\frac{C}{A}Y',
\end{aligned}
$$

and

$$\begin{aligned}\frac{d^2X}{d\tau^2} &= -\,b\sqrt{B^2X'^2+C^2Y'^2}X'\\ \frac{d^2Y}{d\tau^2} &= -\,\frac{A^2}{C}g-b\sqrt{B^2X'^2+C^2Y'^2}Y'.\end{aligned}$$

One reasonable choice is $B=C=\frac{1}{b}$, and $A=\frac{1}{\sqrt{bg}}$. This gives

$$\begin{aligned}\frac{d^2X}{d\tau^2} &= -\,\sqrt{X'^2+Y'^2}X'\\ \frac{d^2Y}{d\tau^2} &= -\,1-\sqrt{X'^2+Y'^2}Y'.\end{aligned}$$

The initial conditions become

$$\begin{aligned}X(0) &= 0\\ Y(0) &= bh_0\\ X'(0) &= \sqrt{\frac{b}{g}}|\vec{v}_0|\cos\theta\\ Y'(0) &= \sqrt{\frac{b}{g}}|\vec{v}_0|\sin\theta.\end{aligned}$$