UNIVERSITY OF CALIFORNIA, SAN DIEGO

On Tensor Categories Arising from Quantum Groups and BMW-Algebras at Odd Roots of Unity

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The dissertation of Eric C. Rowell is approved, and it is acceptable in quality and form for publication on microfilm:

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To Sulem

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ABSTRACT OF THE DISSERTATION

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We consider the premodular (fusion) categories associated to quantum groups corresponding to Lie algebras $\mathfrak{so}_{2k+1}$ of type B and q-Brauer algebras at odd roots of unity. The motivating problem is to determine if the braid group representations on the morphism spaces in these categories are unitarizable for some choice of q. Whereas it was believed that the premodular categories associated to q-Brauer algebras did give rise to unitarizable braid representations, it was only conjectured that this was the case for the quantum group situation. We first prove that these two classes of categories are tensor equivalent. Then we prove the surprising result that the braid group representations are never unitarizable for any choice of q^2 a primitive odd root of unity for *either* of these categories. This result also implies that no C^*-tensor categories exist with the same Grothendieck semiring as these categories. Our computations also allow us to answer the question of modularizability for these categories when the rank k is not divisible by 4.

Chapter 1

Introduction

Let $U_q\mathfrak{g}$ be the quantum group associated to the simple Lie algebra $\mathfrak{g}$ and assume q is a primitive $2l$th root of unity. If we take the usual representation category for the quantum group and specialize at this choice of q, the highest weight representations are no longer irreducible or even semisimple in general. However, using Andersen's [A] category $\mathcal{T}$ of tilting modules over $U_q\mathfrak{g}$ one can remedy the situation by taking a certain semisimple quotient category $\mathcal{F} = \mathcal{T}/\mathcal{I}$ where $\mathcal{I}$ is a tensor ideal. In the general case $\mathcal{F}$ depends on a choice of q and is known to be a premodular category. Several questions arise in connection with these categories. The first of which is a question of classification: How does $\mathcal{F}$ relate to premodular categories constructed by other means? The second question is of interest in the field of operator algebras: When does $\mathcal{F}$ have the structure of a C^*-algebra? The third question is of interest in low-dimensional topology: When is the category modular or, failing that, modularizable?

Kirillov Jr. [Ki] has defined a $*$-operation on the morphisms in $\mathcal{T}$ and conjectured that in certain cases the form $(f, g) = Tr_q(f^*g)$ is positive semidefinite, and therefore the space of morphisms of $\mathcal{F}$ has the structure of a C^*-category. Here Tr_q is the categorical q-trace. This conjecture would imply that for any object $W \in \mathcal{F}$ the representations of Artin's braid group $\mathcal{B}_n$ on $\mathrm{End}_{\mathcal{F}}(W^{\otimes n})$ are unitarizable, since $\mathrm{End}_{\mathcal{F}}(W^{\otimes n})$ is a Hilbert space with the form above. Wenzl [W2] has shown that this indeed is the case for all Lie types for certain choices of q. Specifically, if d is the ratio of square lengths of a short root of $\mathfrak{g}$ to a long one and l is greater than the dual Coxeter number of $\mathfrak{g}$ then the conjecture is true for the choice $q = e^{\pi i/dl}$.

In this paper we will fill out the picture by considering the cases where $\mathfrak{g}$ is of Lie type B and q is a $2l$th root of unity with l odd. That is, we set $q = e^{z\pi i/l}$ with l odd and $\gcd(z, 2l) = 1$. Note that in this situation $d = 2$, which is not covered in [W2]. We answer the classification question by showing that the quotient category $\mathcal{F}$ is tensor equivalent to a tensor category $\mathcal{V}$ derived from certain semi-simple quotients of the BMW-algebras of Wenzl, Birman and Murakami [BW, M]. In a recent paper by Tuba and Wenzl [TuW2], it is shown that the categorical q-dimension on $\mathcal{V}$ is determined up to a sign–assuming there is a braiding. By analyzing the structure of the Grothendieck semiring of $\mathcal{F}$ (or $\mathcal{V}$) we show that our abstract fusion category has a *unique* positive q-character (a generalization of q-dimension), and that there is no choice of q (for l satisfying $2(2k+1) < l$) for which the categorical q-dimension of $\mathcal{V}$ is equal to (plus or minus) the unique positive q-character found. Now having a positive q-dimension is a necessary condition for the form $(\ , \ )$ to be positive semi-definite, as the value of the q-trace on any idempotent is equal to the q-dimension of its image. So we find that there is no C^* structure possible on the category as the morphism spaces in this category are not Hilbert spaces. The modularizability question is a little more hopeful. Using a recent criterion of Brugiéres [Br] we get good results in this direction as well.

Here is a more explicit summary of this paper. Chapter 2 contains mainly definitions that could be found in other papers. In the third chapter we find the unique positive q-character for the Grothendieck semiring of the category $\mathcal{F}$ and an involution of the Grothendieck semiring that preserves q-characters up to a change of sign. This provides us with an invertible object in the category besides the identity. (That is, an object whose equivalence class in the Grothendieck semiring is invertible.) In chapter 4 we briefly discuss the BMW-algebra and associated tensor category in which we are interested. In the fifth chapter we give the proof of equivalence of the categories $\mathcal{F}$ and $\mathcal{V}$. In chapter 6 we give the two main consequences of this equivalence by using the unique positive q-character from chapter 3 to show the failure of positivity, and show that if the rank is not divisible by 4 the category $\mathcal{F}$ is not modularizable. We also mention some ideas for future research.

Chapter 2

Preliminaries

2.1 Quantum Groups

In this section we define what we mean by a quantum group. Our definitions follow [Lu], and proofs can be found in the literature such as [D] and [ChPr].

We begin with the root system Φ for the Lie algebra $\mathfrak{g}$ of rank k with Cartan matrix $A = (a_{ij})$, root basis (or simple roots) $\Pi = \{\alpha_i\}_{i=1}^k$, positive roots $\Phi_+ = \mathbb{N}\Pi \cap \Phi$ and root lattice $Q = \mathbb{Z}\Pi$. We embed the root system in some $\mathbb{R}^n$ and choose a non-degenerate form $\langle\ ,\ \rangle$ on $\mathbb{R}^k$ such that $2\langle \alpha_i, \alpha_j\rangle / \langle \alpha_j, \alpha_j \rangle = a_{ij}$, normalized so that $\langle \alpha, \alpha \rangle = 2$ for short roots. Define the coroot basis $\check{\Pi} = \{\check{\alpha}_i\}_{i=1}^k$ where $\check{\alpha}_i := 2\alpha_i / \langle \alpha_i, \alpha_i \rangle$, so we have a coroot system $\check{\Phi}$ and positive coroots $\check{\Phi}_+$ also embedded in $\mathbb{R}^n$. Let W be the *Weyl* group generated by the reflections on $\mathbb{R}^n$: $s_i(v) := v - \langle v, \check{\alpha}_i \rangle \alpha_i$. Although we have identified the roots and coroots with vectors in $\mathbb{R}^n$, we usually think of the coroots as elements of the Cartan subalgebra $\mathfrak{h}$ and the roots as linear functionals on $\mathfrak{h}$ via the form $\langle\ ,\ \rangle$.

Fix a formal variable q and define $q_i = q^{d_i}$ where $d_i = \langle \alpha_i, \alpha_i \rangle / 2$. Denote the q-number $\frac{q^n - q^{-n}}{q - q^{-1}}$ by $[n]$, and let $[n]_i$ represent the same formula with q replaced by q_i. Now define $\mathcal{A}(\mathfrak{g})_q$ to be the $\mathbb{R}(q)$-algebra with generators $E_i, F_i, h_i, h_i^{-1} (1 \leq i \leq k)$

and relations:

$$
\begin{aligned}
&[h_i^{\pm 1}, h_j^{\pm 1}] = 0, \qquad h_i h_i^{-1} = 1,\\
&h_i E_j h_i^{-1} = q^{a_{ij}} E_j, \quad h_i F_j h_i^{-1} = q^{-a_{ij}} F_j,\\
&[E_i, F_j] = \delta_{ij} \frac{h_i^{d_i} - h_i^{-d_i}}{q_i - q_i^{-1}},\\
&\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_i E_i^{1-a_{ij}-t} E_j E_i^t = 0, \quad i \neq j \text{ (quantum Serre relation)}
\end{aligned}
$$

where

$$\begin{bmatrix} a \\ b \end{bmatrix}_i = \frac{[a]_i!}{[b]_i![a-b]_i!}$$

denotes the q_i-binomial coefficient. A similar quantum Serre relation holds among the F_i.

2.1.1 Hopf Algebra Structure

Next we define $H_i = h_i^{d_i}$, and if $\mu = \sum m_i \alpha_i \in Q$ (m_i integers) then $H_\mu = \prod H_i^{m_i}$. $\mathcal{A}(\mathfrak{g})_q$ has the following Hopf algebra structure defined on generators:

Comultiplication:

$$
\begin{aligned}
\Delta(E_i) &= H_i \otimes E_i + E_i \otimes 1,\\
\Delta(F_i) &= 1 \otimes F_i + F_i \otimes H_i^{-1},\\
\Delta(h_i) &= h_i \otimes h_i
\end{aligned}
$$

Antipode:

$$
\begin{aligned}
S(E_i) &= -H_i^{-1} E_i,\\
S(F_i) &= -F_i H_i,\\
S(h_i) &= h_i^{-1}
\end{aligned}
$$

Counit:

$$
\begin{aligned}
\epsilon(E_i) &= \epsilon(F_i) = 0,\\
\epsilon(h_i) &= 1.
\end{aligned}
$$

These structures satisfy a number of compatibility conditions that can be found in any book on Hopf algebras. Furthermore $\mathcal{A}(\mathfrak{g})_q$ is quasitriangular, which means that there is an invertible element R called the *universal R-matrix* in some completion of $\mathcal{A}(\mathfrak{g})_q \otimes \mathcal{A}(\mathfrak{g})_q$ that intertwines the comultiplication and the opposite comultiplication and satisfies certain other compatibility conditions. The importance of the R-matrix will be discussed in Section 2.5. Explicit formulas for the universal R-matrix can be found in [Lu].

Now we define $U_q\mathfrak{g}$ to be the quasitriangular Hopf subalgebra of $\mathcal{A}(\mathfrak{g})_q$ generated by the *divided powers* $E_i^{(p)} = E_i^p/[p]_i!$, $F_i^{(p)} = F_i^p/[p]_i!$ (where $p \geq 1$) and the $h_i^{\pm 1}$, with the relations induced from $\mathcal{A}(\mathfrak{g})_q$. $U_q\mathfrak{g}$ is now well-suited for specialization to q a root of unity, and the R-matrix specializes to $U_q\mathfrak{g}$ as well.

2.1.2 Type B Data

Now we restrict our attention to $\mathfrak{g}$ of type B; that is, $\mathfrak{g} = \mathfrak{so}_{2k+1}$. Let $\{\varepsilon_i\}$ be the standard basis for $\mathbb{R}^k$. We fix a root basis

$$\Pi = \{\alpha_i\}_1^k = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_k - \varepsilon_{k+1}, \varepsilon_k\}$$

so the root lattice $Q = \mathrm{span}_{\mathbb{Z}}\{\mathrm{\alpha_i}\}_1^{\mathrm{k}}$ is just $\mathbb{Z}^k$. We also record that the set of positive roots is

$$\Phi_+ = \{\varepsilon_s \pm \varepsilon_t, \varepsilon_u : s < t\}.$$

The form $\langle\ ,\ \rangle$ is twice the usual dot product on $\mathbb{R}^k$ so that the square length of long roots is 4, and 2 for short roots. Thus the coroot basis $\check{\Pi} = \{\check{\alpha}\}$ has

$$\check{\alpha}_i = \begin{cases} \frac{1}{2}(\varepsilon_i - \varepsilon_{i+1}) & i = 1, \ldots, k-1 \\ \varepsilon_k & i = k \end{cases}$$

Note that the coroots for type B are the roots of type C in the classical Lie algebra case, but here we must take care as the normalization of the form is not the classical one. We will see where this leads to subtitles later. The Weyl group W is the semidirect product of S_k and $(\mathbb{Z}_2)^k$ and acts on $\mathbb{R}^k$ via permutations and sign changes. Throughout this paper we will denote $U_q\mathfrak{so}_{2k+1}$ simply by U.

2.2 Representations of U

For our choice of a root basis we have the following *fundamental weights*:

$$\Lambda_i = \begin{cases} \sum_{1\le j\le i-1} \varepsilon_j & i \le k-1 \\ \frac{1}{2}\sum_{1\le j\le k} \varepsilon_i & i = k \end{cases}$$

and the *dominant weights* $P_+ = \mathrm{span}_{\mathbb{N}}\{\Lambda_j\}_1^k$ which reside in the dominant Weyl chamber. Note that $\langle \Lambda_i, \check{\alpha}_j\rangle = \delta_{ij}$, so that the fundamental weights are dual to the coroots. The weight lattice $P = \mathrm{span}_{\mathbb{Z}}\{\Lambda_j\}_1^k$ is then seen to be $\mathbb{Z}^k \bigcup (\Lambda_k + \mathbb{Z}^k)$. For convenience of notation we introduce the function on P:

$$p(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \mathbb{Z}^k \\ -1 & \text{if } \lambda \in (\Lambda_k + \mathbb{Z}^k) \end{cases}$$

We refer to a weight λ as integral, resp. half-integral, if $p(\lambda) = 1$, resp. $p(\lambda) = -1$. The weights are usually represented as k-tuples, e.g. $\Lambda_k = (1/2, \ldots, 1/2)$. There is a one-to-one correspondence between the dominant weights and finite-dimensional, simple, integrable U-modules. A finite-dimensional integrable U-module is a $\mathbb{C}$-vector space V satisfying the following:

$$V = \bigoplus_{\lambda\in P} V^\lambda, \qquad h_i|_{V^\lambda} = q^{\langle \check{\alpha}_i, \lambda\rangle}\mathbf{1}_{V^\lambda}$$

$$E_i^{(p)}(V^\lambda) \subset V^{\lambda + p\alpha_i}, \qquad F_i^{(p)}(V^\lambda) \subset V^{\lambda - p\alpha_i}$$

Since these are the only modules we will consider, we will just refer to them as U-modules with this understanding. The action of U on such a module is still well-defined when we specialize q at any nonzero complex number. Furthermore, the dual of a simple module V_λ is defined via the antipode. It is denoted V_λ^* and has highest weight $-w_0(\lambda)$ where w_0 is the longest element of the Weyl group. For type B we have that $w_0 = -1$, that is, the element of the Weyl group that changes the sign of each coordinate. So we have $V_\lambda \cong V_\lambda^*$ in the present case. We also have a trivial one-dimensional module where the action of U is by the counit. Using the comultiplication, we can define an action of U on the tensor product of any two U-modules V and W by $\Delta(u)(v \otimes w)$ for $u \in U$, $v \in V$ and $w \in W$. The compatibility conditions of the Hopf algebra structure imply that the tensor product is associative if we identify isomorphic modules. Furthermore, the action of the R-matrix is well-defined on any tensor product of two U-modules as all but finitely many of its terms act as zero on such a tensor product (see [Lu] for details). More will be said about this in Section 2.5. It is well-known (see [BK]) that for $|q| \neq 1$ (the generic case) the tensor product rules for U are the same as those for the classical $U\mathfrak{so}_{2k+1}$ representation category. A representation category for a Hopf algebra is the category whose objects are a set of modules closed under the tensor product and direct sums, and the morphisms are the intertwining operators among them. Our categories will always include the trivial module and be closed under the operation of taking dual modules.

2.3 Classical Representation Theory, Abridged

For generic q the representation (tensor) category of U can be understood from the classical theory, so in this section we will summarize the necessary facts from the representation theory of the algebra $U\mathfrak{so}_{2k+1}$ and the Lie group $O(2k+1)$ which will be used in later chapters. This material can be found in any introductory text on Lie groups, such as [GWa] or [Hu], and goes back at least to Weyl [Wy].

2.3.1 The Lie Algebra $\mathfrak{so}_{2k+1}$

As we observed above, the irreducible finite-dimensional integral highest weight modules of $U\mathfrak{so}_{2k+1}$ are in one-to-one correspondence with the elements of P_+. Each irreducible representation V_λ has a multiset of weights $P(\lambda)$ which correspond to the weight-space decomposition of V_λ with respect to the action of the Cartan subalgebra. The multiset $P(\lambda)$ lies in the ball of radius $|\lambda|$ (ordinary euclidian distance) centered at the origin, and the weights in the W-orbit of λ appear with multiplicity one. The other weights are of the form $\lambda - \alpha$ for some $\alpha \in Q$. To decompose the tensor product of two irreducible modules V_λ and V_μ one looks at the intersection $\{\nu = \mu + \kappa : \kappa \in P(\lambda)\} \bigcap P_+$ which contains the dominant weights of the irreducible submodules

$$P_+(V_\lambda \otimes V_\mu) = \{\nu \in P_+ : V_\nu \subset V_\lambda \otimes V_\mu\}.$$

We do not formulate the precise algorithm to determine which V_ν do occur nor the multiplicities, but we can say that the irreducible module $V_{\mu+w(\lambda)}$ appears with multiplicity one, where w is any element in the Weyl group such that $w(\lambda)+\mu \in P_+$. (This follows from the outer multiplicity formula, see e.g. [GWa] Corollary 7.1.6). Moreover, $P_+(V_\lambda \otimes V_\mu)$ is contained in the ball of radius $|\lambda|$ centered at μ, and $p(\nu) = p(\lambda)p(\mu)$ for any $\nu \in P_+(V_\lambda \otimes V_\mu)$. In other words, all weights of simple submodules of $V_\lambda \otimes V_\mu$ are integral if λ and μ are both integral or half-integral, and half-integral otherwise. We write: $V_\lambda \otimes V_\mu = \bigoplus_\nu m^\nu_{\lambda\mu} V_\nu$ where $m^\nu_{\lambda\mu}$ is the multiplicity of V_ν in $V_\lambda \otimes V_\mu$.

2.3.2 The Lie Group $O(2k+1)$

The irreducible representations of the compact group $O(2k+1)$ are labelled by Ferrers diagrams with at most $2k+1$ boxes in the first two columns. The identity component of $O(2k+1)$ is $SO(2k+1)$, so the Lie algebra of either group is $\mathfrak{so}_{2k+1}$. The representations of $SO(2k+1)$ are the integral weight representations of $\mathfrak{so}_{2k+1}$, and any irreducible representation of $O(2k+1)$ is determined by the action of $-I$ (which is not in $SO(2k+1)$) and the restriction to $SO(2k+1)$. The restriction rules from $O(2k+1)$ to $SO(2k+1)$ go as follows: for λ any Ferrers diagram with at most $2k+1$ boxes in the first two columns, define $\overline{\lambda}$ to be the diagram with $\min\{2k+1-\lambda'_1, \lambda'_1\}$ boxes in the first column, where λ'_1 is the number of boxes in the first column of λ. So $\overline{\lambda}$ will have at most k rows, and by filling in zeros if needed, $\overline{\lambda}$ can be made into a dominant $SO(2k+1)$ weight by writing $(\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_k)$ where

$\overline{\lambda}_i$ is the number of boxes in the ith row of the diagram $\overline{\lambda}$. Then the restriction of the irreducible $O(2k+1)$-module W_λ to $SO(2k+1)$ is the irreducible module $V_{\overline{\lambda}}$. Now the action of $-I$ on any irreducible module W_λ is just $(-1)^{|\lambda|}$, so we have the following algorithm for decomposing the tensor product of two irreducible $O(2k+1)$-modules W_λ and W_μ. One decomposes the $SO(2k+1)$-module $V_{\overline{\lambda}} \otimes V_{\overline{\mu}}$ into irreducible submodules V_{ν_i} and takes the corresponding $O(2k+1)$-modules whose labels (Ferrers diagrams) have the same parity as $|\lambda| + |\mu|$.

2.4 Tilting Modules

In this section we develop some of the framework necessary to obtain the categorical structures the definitions of which we will postpone until Chapter 3. To describe the modifications necessary to obtain a semisimple tensor category from $Rep(U)$ at a root of unity, we begin with the Weyl modules discussed in Section 2.1. Consider the q generic case for a moment. For every $\lambda \in P_+$ there is a unique irreducible Weyl module V_λ. The module corresponding to weight $\mathbf{0}$ is the trivial module $V_0 = \mathbb{1}$. There exists a basis so that the action of the generators of U is given by matrices with entries in $\mathbb{Z}[q, q^{-1}]$ and in particular is well-defined for any nonzero complex q. The Weyl module V_{Λ_k} is called the fundamental module, as all Weyl modules appear in some tensor power of V_{Λ_k}.Now assume q is a primitive $2l$th root of unity l odd; so that $q^l = -1$. Later we will see that the case where $q^l = 1$ is covered as well, as the change of variables $q \to -q$ does not change the basic structure. We define a *tilting module* to be a U-module T that is a direct summand of some tensor power of V_{Λ_k} or a direct sum of such modules. The set of tilting modules $\mathcal{T}$ can be given the structure of a ribbon category. We will describe this in more detail later. For any U-module W we define the categorical q-trace Tr_q on $\mathrm{End}(W)$ by $Tr_q(f) = Tr(fH_{2\rho})$, where $H_{2\rho}$ is a canonical element of U associated with the quasi-triangularity of the quantum group and the so-called "quantum Casimir" (see [Lu]). Here 2ρ is the sum of the positive roots. There is an equivalent definition for the trace which can be described entirely in the language of categories and uses the ribbon structure on $\mathcal{T}$, which will be defined in Section 3.1. This q-trace gives rise to a categorical q-dimension defined by $\dim_q(T) = Tr_q(\mathbf{1}_T)$ where $\mathbf{1}_T$ is the identity morphism on T. For a Weyl module V_λ and a fixed $q \in \mathbb{C} - \{0, \pm 1\}$ the q-dimension takes the form:

$$\dim_q(V_\lambda) = \sum_{w \in W} \varepsilon(w) q^{2\langle w(\lambda+\rho), \rho\rangle} = \prod_{\alpha \in \Phi_+} \frac{[\langle \lambda + \rho, \alpha\rangle]}{[\langle \rho, \alpha\rangle]}.$$

A proof of this formula can be found in [A] for quantum groups. The canonical q-dimension function $\dim_q$ has two important properties (also shown in [A]):

$$\dim_q(\mathbb{1}) = 1 \tag{2.1}$$

$$\dim_q(T_1 \otimes T_2) = \dim_q(T_1) \dim_q(T_2) \tag{2.2}$$

Property 2.2 is the critical property and will be mentioned again in the context of q-characters in Section 3.4.

The q-dimension gives us a way of recovering semisimplicity, as the set of tilting modules T with $\dim_q(T) = 0$ forms a tensor ideal (see the properties of tilting modules below). Such modules are called *negligible*. To facilitate this program we must extend the Weyl group as follows.

Definition 2.1. Consider the affine reflection in $\mathbb{R}^k$ through the hyperplane $\{x \in \mathbb{R}^k : \langle x, \varepsilon_1\rangle = l\}$. If we adjoin this reflection t_l to the Weyl group W we get the *affine* Weyl group W_l. Explicitly $t_l(\lambda) = \lambda + (l - \langle \lambda, \varepsilon_1\rangle)\varepsilon_1$.

Each element w of W_l can be expressed as a product of simple reflections s_i and t_l, and furthermore any decomposition of w into a product of simple reflections will always have the same number of terms modulo 2. Thus the usual sign function $\varepsilon(w)$ is well-defined if we require $\varepsilon(t_l) = -1$. In addition to the usual action of W, we will need the modified "dot" action of W_l on the weight lattice: $w \cdot \lambda = w(\lambda+\rho) - \rho$. For example,

$$t_l \cdot \lambda = \lambda + \rho + (l - \langle \lambda + \rho, \varepsilon_1\rangle)\varepsilon_1 - \rho = \lambda + (l - \langle \lambda + \rho, \varepsilon_1\rangle)\varepsilon_1$$

and for a simple reflection s_i we have

$$s_i \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha_i \rangle \alpha_i$$

Let D_0 be the closure of the region in $\mathbb{R}^k$ containing the origin bounded by the hyperplanes $\mathcal{H}_i, \mathcal{H}_l$ fixed under the dot action of W_l, so that D_0 is a fundamental domain for this action. For l odd define the *fundamental alcove* C_l as the weight lattice P intersected with the interior D_0, that is, $C_l = \{\lambda \in P_+ : \langle \lambda + \rho, \varepsilon_1 \rangle < l\}$. We also define $\overline{C_l} = P_+ \cap D_0 = \{\lambda \in P_+ : \langle \lambda + \rho, \varepsilon_1 \rangle \leq l\}$. Explicitly we have

$$C_l = \{\lambda \in \mathbb{Z}^k \cup (\mathbb{Z}^k + \Lambda_k) : \frac{l - 2k}{2} \geq \lambda_1 \geq \lambda_2 \geq, \ldots, \geq \lambda_k \geq 0\}$$

and we can compute $|C_l| = 2\binom{\frac{l-1}{2}}{k}$. Observe that any μ in P_+ is conjugate via W_l to a unique $\kappa \in D_0$ and that if $w \cdot \mu = \kappa$ with $\kappa \in C_l$ the element w is also unique. We will always require (as other authors do) that the tilting module with weight $\rho + \Lambda_k$ is in the fundamental alcove. This puts a lower bound on l in terms of the rank k: $l \geq 4k+1$. The category $\mathcal{T}$ of tilting modules have the following important properties (see [A] and [AP]):

1. Any tensor product of tilting modules is again a tilting module.
2. For any dominant weight $\mu \in P_+$ there is a unique indecomposable tilting module T_μ.
3. $T_\mu = V_\mu$ (a Weyl module) for any $\mu \in \overline{C_l}$, and these modules are simple.
4. Every tilting module is a direct sum of indecomposable tilting modules.
5. Every $T \in \mathcal{T}$ can be expressed as $T = C_T \oplus C'_T$ where C_T is a sum of indecomposable modules T_μ with $\mu \in C_l$ and C'_T is a sum of indecomposable modules T_κ with $\kappa \in P_+ \setminus C_l$.
6. $\dim_q T_\lambda = 0$ if and only if $\lambda \in P_+ \setminus C_l$.
7. The set of negligible modules $\mathcal{I} = \{T \in \mathcal{T} : \dim_q T = 0\}$ is a tensor ideal in $\mathcal{T}$.

8. (See Chapter 3 for definitions) $\mathcal{F} = \mathcal{T}/\mathcal{I}$ is a semisimple, ribbon Ab-category with simple objects labelled by dominant weights $\lambda \in C_l$.

9. Every simple object in $\mathcal{F}$ appears as a subobject of a tensor power of the fundamental module V_{Λ_k}.

The interested reader should see the references above for proofs, although we make the following remarks:

Remark 2.2. To see part of property 6 note that if $\lambda \in \overline{C_l} \setminus C_l$, that is, $\langle \lambda+\rho, \varepsilon_1 \rangle = l$ then the factor $[\langle \lambda + \rho, \varepsilon_1 \rangle]$ of the numerator of $\dim_q(V_\lambda)$ vanishes for q any $2l$th root of unity. (See property 3 above.) Since $(\lambda)_1 \geq (\lambda)_2 \geq \ldots \geq (\lambda)_k$, we further observe that if $\lambda \in C_l$ and $\alpha = \varepsilon_s \pm \varepsilon_t$ is a long root, $\langle \lambda+\rho, \alpha \rangle \leq \langle \lambda+\rho, \varepsilon_1+\varepsilon_2 \rangle < 2l$ and is even, whereas if $\alpha = \varepsilon_u$ is a short root, $\langle \lambda + \rho, \alpha \rangle \leq \langle \lambda + \rho, \varepsilon_1 \rangle < l$. Now for q a primitive $2l$th root of unity, the q-number $[n] = 0$ if and only if $l|n$. Thus one sees that $\dim_q$ vanishes on $\overline{C_l} \setminus C_l$ and is non-zero on C_l.

Remark 2.3. Property 7 above is crucial: it says that $V_\lambda \in \mathcal{I}$ if $\lambda \in \overline{C_l} \setminus C_l$, and by virtue of property 1 any submodule of $V_\lambda \otimes T$ also has q-dimension 0. The set of V_λ with $\lambda \in \overline{C_l} \setminus C_l$ in fact generates the ideal $\mathcal{I}$. For, every tilting module is a submodule of $V_{\Lambda_k}^{\otimes n}$ or a sum of such submodules and thus by tensoring some negligible V_λ with an appropriate tensor power $V_{\Lambda_k}^{\otimes n}$ we can get every module in the ideal $\mathcal{I}$. This shows that *every* tilting module T_μ with $\mu \notin C_l$ is in $\mathcal{I}$.

The quotient (functor) map of $\mathcal{T}$ onto $\mathcal{F}$ is defined by projecting each $T \in \mathcal{T}$ orthogonally onto its component C_T which is a direct sum of simple Weyl modules by the third property. The simple objects of $\mathcal{F}$ are the images of these simple Weyl modules V_μ with $\mu \in C_l$, and we will abuse notation by continuing to denote them by V_μ. Since C_l is a finite set, there are finitely many simple objects in $\mathcal{F}$. The image of a morphism $f : T_1 \to T_2$ under the quotient map is (the class of) the morphism $p_2 \circ f \circ p_1 : C_{T_1} \to C_{T_2}$, where p_i is the orthogonal projection of T_i onto C_{T_i} with kernel C'_{T_i}. We denote the space of morphisms between two objects in $\mathcal{F}$ by $\mathrm{Hom}_{\mathcal{F}}(C_{T_1}, C_{T_2})$.

The tensor product rules (or fusion rules) on $\mathcal{F}$ can now be described as follows:

Proposition 2.4. *[AP] Let $m_{\lambda\mu}^{\nu}$ be the multiplicity of the simple module V_ν in $V_\lambda \otimes V_\mu$ considered as $\mathfrak{so}_{2k+1}$-modules. Then as objects in $\mathcal{F}$ one has:*

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu \in C_l} N_{\lambda\mu}^{\nu} V_\nu$$

where

$$N_{\lambda\mu}^{\nu} = \sum_{W_\nu} \varepsilon(w) m_{\lambda\mu}^{w\cdot(\nu)}$$

and $W_\nu = \{w \in W_l : w \cdot \nu \in P_+\}$.

Notice that if ν is any element of P on the boundary of D_0 all $m_{\lambda\mu}^{w\cdot\nu}$ will cancel out, since both w and $s_i w$ (or $t_l w$) will be in W_ν and have opposite signs. Notice also that $p(\lambda) = p(w \cdot \lambda)$ for all $w \in W_l$, that is, the dot action carries integral weights to integral weights and half-integral weights to half-integral weights. Since Weyl modules are self-dual (for $\mathfrak{so}_{2k+1}$), and the tensor product in $\mathcal{F}$ is commutative (via the braiding) the following shows that the $N_{\lambda\mu}^{\nu}$ (the *fusion coefficients*) are symmetric in all three indices:

$$N_{\lambda\mu}^{\nu} = \dim \operatorname{Hom}_{\mathcal{F}}(V_\nu, V_\lambda \otimes V_\mu) = \dim \operatorname{Hom}_{\mathcal{F}}(\mathbb{1}, V_\lambda \otimes V_\mu \otimes V_\nu^*).$$

In general it is not easy to compute the $N_{\lambda\mu}^{\nu}$ as it is already difficult to compute the classical multiplicities $m_{\lambda\mu}^{\nu}$; however, for our analysis we only require two explicit decomposition rules–both of which were already known to Brauer. We begin with the decomposition rules for tensoring with the generating module V_{Λ_k}.

Example 2.5. We have that V_{Λ_k} is a minuscule representation (all weights are conjugate under the Weyl group) the simple decomposition as a $\mathfrak{so}_{2k+1}$-module is:

$$V_{\Lambda_k} \otimes V_\lambda = \bigoplus_{W_k^\lambda} V_{\lambda + w(\Lambda_k)}$$

where $W_k^\lambda = \{w \in W : \lambda + w(\Lambda_k) \in P_+\}$ Note that $W(\Lambda_k) = \{\frac{1}{2}(\pm 1, \ldots, \pm 1)\}$, so all $\lambda + w(\Lambda_k)$ are in $\overline{C_l}$, and the decomposition in our quotient category $\mathcal{F}$ is gotten by discarding the $V_{\lambda+w(\Lambda_k)} \in \overline{C_l} \setminus C_l$. That is, for $\lambda, \nu \in C_l$

$$N^{\nu}_{\Lambda_k \lambda} = \begin{cases} 1 & \text{if } \nu = \lambda + w(\Lambda_k) \text{ some } w \in W \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

By the following arguement we see that there exists an odd integer s such that *every* simple object in $\mathcal{F}$ appears in $V_{\Lambda_k}^{\otimes s}$ or $V_{\Lambda_k}^{\otimes s+1}$. Every weight $\lambda \in C_l$ can be expressed as a sum of weights in $W(\Lambda_k)$, so every V_λ appears in some tensor power of V_{Λ_k} by an induction using the multiplicity formula above. Furthermore, the trivial representation $\mathbb{1}$ appears in $V_{\Lambda_k}^{\otimes 2}$ so once V_λ appears in an odd (resp. even) tensor power of V_{Λ_k} it will appear in *every* odd (resp. even) tensor power thereafter.

The vector (or defining) representation of $\mathfrak{so}_{2k+1}$ (as well as that of U) has highest weight $\Lambda_1 = \varepsilon_1$. We will only need to know the complete decomposition for tensoring V_{Λ_1} with simple objects whose highest weight has integer entries as follows:

Example 2.6. The weights of V_{Λ_1} are the zero weight together with $W(\Lambda_1) = \{\pm\varepsilon_i : 1 \leq i \leq k\}$. The decomposition algorithm as a $\mathfrak{so}_{2k+1}$-module is (for integral weights μ):

$$V_{\Lambda_1} \otimes V_\mu = \delta(\mu) V_\mu + \bigoplus_{W_1} V_{\mu+w(\Lambda_1)}$$

where $W_1 = \{w \in W : w(\Lambda_1) \in P_+\}$ and $\delta(\mu) = 1$ if $\langle \mu, \alpha_k \rangle > 0$ and zero otherwise. Since the dominant weights on the hyperplane $\mathcal{H}_l$ all have integer entries and μ is distance at least 1 from $\mathcal{H}_l$, we conclude that $P_+ \cap (\mu + W(\Lambda_1)) \subset \overline{C_l}$. So using the proposition above we see that as objects in the category $\mathcal{F}$ the decomposition is gotten by discarding those $V_{\mu+w(\Lambda_1)}$ with $\mu + \Lambda_1$ on $\mathcal{H}_l$. So for $\mu, \nu \in C_l \cap \mathbb{Z}^k$ we compute:

$$N^{\nu}_{\Lambda_1 \mu} = \begin{cases} 1 & \text{if } \nu = \mu \pm \varepsilon_i \text{ for some } 1 \leq i \leq k \\ 1 & \text{if } \mu = \nu \text{ and } \langle \mu, \varepsilon_k \rangle > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

The various categories $\mathcal{F}$ have slightly different properties depending on the root of unity q. For example, if $q = e^{z\pi i/l}$ with z odd the q-dimension function defined above becomes:

$$\dim_q(V_\lambda) = \prod_{\alpha \in \Phi_+} \frac{\sin(\langle \lambda + \rho, \alpha \rangle z\pi i/l)}{\sin(\langle \rho, \alpha \rangle z\pi i/l)}.$$

Observe that for any z there are choices of λ and $\alpha \in \Phi_+$ such that $\langle \lambda + \rho, \alpha \rangle > l$, so there is no guarantee that the q-dimension is positive on C_l. In fact we will see later that there is no choice of z such that $\dim_q$ is positive for all $\lambda \in C_l$.

2.5 Action of $\mathcal{B}_n$ on $\mathrm{End}_{\mathcal{F}}(W^{\otimes n})$

Artin's braid group $\mathcal{B}_n$ is the group on $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying the two relations:

1. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

2. $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| \geq 2$.

A typical generator for the topological presentation of $\mathcal{B}_n$ is given in Figure 2.1.

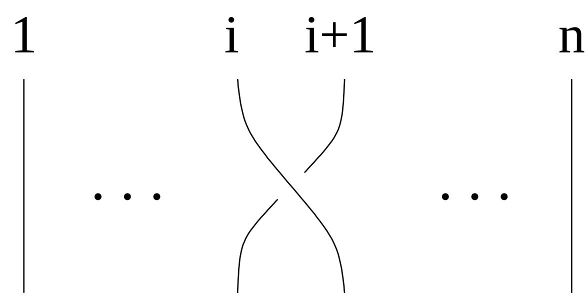

Figure 2.1: The generator σ_i

The fact that this presentation is isomorphic to the topological braid group generated by n arcs strung between two rows of n points is the celebrated theorem of Artin.

In Section 2.1 we mentioned an element R in a completion of $U \otimes U$ that intertwines the coproduct and the opposite coproduct. From this element we can define for any two U-modules V and W a U-module isomorphism $\check{R}_{VW} : V \otimes W \to W \otimes V$ by

$$\check{R}_{VW}(v_1 \otimes w_1) = P_{VW} R(v_1 \otimes w_1)$$

where P_{VW} interchanges the factors in $V \otimes W$. This shows that the tensor product is commutative on isomorphism classes of modules. We are most interested in the case where $V = W$, for then the fact that R is a solution to the Yang-Baxter equation gives us representations of $\mathcal{B}_n$ as follows. We denote $\check{R}_{WW}$ by $\check{R}^W$. Define $R_i = \mathbf{1}_{i-1} \otimes \check{R}^W \otimes \mathbf{1}_{n-i-1} \in \mathrm{End}_{\mathcal{F}}(W^{\otimes n})$ for $i = 1, \ldots, n-1$ where $\mathbf{1}_j$ is the identity on $W^{\otimes j}$ and $\check{R}^W$ acts on the ith and $(i+1)$st components. Then the R_i satisfy $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$ and hence $\sigma_i \to R_i$ defines a representation of the braid group.

From the following result one computes the eigenvalues of $\check{R}_{V_\lambda V_\mu}$ on irreducible submodules (see [D]):

Proposition 2.7 (Drinfeld). *Let V_λ and V_μ be Weyl modules of U. Define for any $\gamma \in P_+$ the number*

$$c_\gamma = \langle \gamma + 2\rho, \gamma \rangle \in \frac{1}{2}\mathbb{Z}.$$

Then the restriction of $\check{R}_{V_\lambda V_\mu}$ to an irreducible submodule is given by:

$$\check{R}_{V_\lambda V_\mu} |_{V_\nu \subset V_\lambda \otimes V_\mu} = q^{c_\nu - c_\lambda - c_\mu} \mathbf{1}_{V_\nu}.$$

Now fix $W \in \mathcal{F}$ and n. The above action of $\mathcal{B}_n$ induces an action on $\mathrm{End}_{\mathcal{F}}(W^{\otimes n})$ via $\sigma_i \cdot f = R_i f$. It can be shown that these are unitary representations precisely when $\mathrm{End}_{\mathcal{F}}(W^{\otimes n})$ is a Hilbert space with respect to the Hermitian form $(f, g) = Tr_q(f^* g)$ (see [W2]). This, of course, depends on the particular choice of the parameter q–the Hilbert space axiom that often fails is the positivity of $(\ ,\ )$.

Chapter 3

Categorical Definitions

In this chapter we briefly discuss the categorical structure of $\mathcal{F}$. All of the terms and definitions used here can be found in great detail in the books by Kassel [K] and Turaev [Tur].

3.1 Premodular Categories

The most specific structure we can put on the category $\mathcal{F}$ is the structure of a *premodular* category (to be defined shortly). Many authors (e.g. [Tur]) prove that when q is a $2l$th root of unity with l even, Andersen's quotient category is modular. The only place the parity of l is used is in proving the S-matrix (see Section 3.2) is invertible. In order to give a complete definition of a premodular category we would need to define a number of axioms. We will describe a few important ones and leave the interested reader to find a full treatment in the references mentioned above.

Definition 3.1. A premodular category is a semisimple ribbon Ab-category that has finitely many isomorphism classes of simple objects.

Semisimplicity means that the tensor product of any two objects in the category can be decomposed into a sum of simple constituent objects. A ribbon Ab-category

$\mathcal{O}$ has duality, a braiding and a twist defined for all $X, Y, Z \in Ob(\mathcal{O})$:

1. **Duality**: There is a module X^* for each object X and maps

$$b_X : \mathbb{1} \to X \otimes X^*, d_X : X^* \otimes X \to \mathbb{1}$$

satisfying

$$\begin{aligned} (\mathbf{1}_X \otimes d_X)(b_X \otimes \mathbf{1}_X) &= \mathbf{1}_X && (3.1) \\ (d_X \otimes \mathbf{1}_{X^*})(\mathbf{1}_{X*} \otimes b_X) &= \mathbf{1}_{X^*}. && (3.2) \end{aligned}$$

The morphisms b_X and d_X are often called *rigidity morphisms.*

2. **Braiding**: Natural isomorphisms

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying

$$\begin{aligned} c_{X,Y\otimes Z} &= (\mathbf{1}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathbf{1}_Z) && (3.3) \\ c_{X\otimes Y,Z} &= (c_{X,Z} \otimes \mathbf{1}_Y)(\mathbf{1}_X \otimes c_{Y,Z}) && (3.4) \end{aligned}$$

3. **Twist**: Natural isomorphisms

$$\theta_X : X \to X$$

such that

$$\begin{aligned} \theta_{X\otimes Y} &= c_{Y,X} c_{X,Y} (\theta_X \otimes \theta_Y) && (3.5) \\ \theta_{X^*} &= (\theta_X)^* && (3.6) \end{aligned}$$

Remark 3.2. The braiding isomorphisms give rise to representations of the braid groups $\mathcal{B}_n$ on the tensor powers of objects in $\mathcal{O}$ as we indicated in for the quantum group case in Section 2.5. In fact, there are several braiding structures on a given braided category generated by an object X–for then all braiding morphisms are determined by $c_{X,X}$, which can be replaced by $-c_{X,X}$, $c_{X,X}^{-1}$ or $-c_{X,X}^{-1}$ to get other braidings.

Remark 3.3. Aside from the braiding, the importance of the ribbon structure (for our purposes) is that it gives us a purely categorical way to define the trace of any endomorphism. It is defined as follows for an endomorphism $f \in \mathrm{End}_{\mathcal{O}}(X)$:

$$tr_{\mathcal{O}}(f) = d_X c_{X,X^*}((\theta_X f) \otimes \mathbf{1}_{X^*}) b_X : \mathbb{1} \to \mathbb{1}.$$

It can be shown (see e.g. [OW] Proposition 1.4) that any premodular category satisfies the *Markov property*, that is, if $a \in \mathrm{End}_{\mathcal{O}}(X^{\otimes n})$ and $m \in \mathrm{End}_{\mathcal{O}}(X^{\otimes 2})$ then

$$tr_{\mathcal{O}}((a \otimes \mathbf{1}_X) \circ (\mathbf{1}_X^{\otimes(n-1)} \otimes m)) = tr_{\mathcal{O}}(a) tr_{\mathcal{O}}(m).$$

Remark 3.4. The categorical trace $tr_{\mathcal{F}}$ is the same as the trace on $\mathcal{F}$ induced from Tr_q of Section 2.4. The braiding in $\mathcal{F}$ is precisely the U-module isomorphisms $\check{R}_{VW}$ described in Section 2.5. The twist in $\mathcal{F}$ is given by multiplying by the quantum Casimir element Θ defined in [Lu]. The action of Θ on a simple Weyl module V_λ is by $q^{\langle \lambda + 2\rho, \lambda \rangle} \mathbf{1}_{V_\lambda}$.

3.2 The S-Matrix

Let $X_1, \ldots, X_t$ be representatives of the finitely many isomorphism classes of simple objects in a premodular category $\mathcal{O}$. The *S-matrix* is defined as:

$$(S_{i,j}) = tr_{\mathcal{O}}(c_{X_i,X_j} \circ c_{X_j,X_i}).$$

A premodular category is said to be modular if S is invertible. The invertibility of the S-matrix provides a representation of the modular group, $SL(2, \mathbb{Z})$ on the t-dimensional vector space with a basis labelled by the simple objects in $\mathcal{O}$. It is shown in e.g. [TW1] that the entries of the S-matrix for the category $\mathcal{F}$ is:

$$S_{\lambda\mu} = \sum_{w \in W} \varepsilon(w) q^{2\langle w(\lambda+\rho), \mu+\rho \rangle} \tag{3.7}$$

where λ and $\mu \in C_l$. The form $\langle \ , \ \rangle$ is invariant under the action of W, so the S-matrix is symmetric. We shall see later that the normalized columns $(\tilde{S}_{\lambda\mu})_\lambda = (S_{\lambda\mu}/S_{\mathbf{0}\mu})_\lambda$ of the S-matrix are q-characters. In fact, the quantity $\dim_q(V_\lambda)$ defined in Section 2.4 is equal to $\tilde{S}_{\lambda\mathbf{0}}$ for all $\lambda \in \overline{C_l}$, that is, for Weyl modules.

3.3 The Grothendieck Ring

The Grothendieck semiring associated to a semisimple ribbon category $\mathcal{O}$ is just the underlying ring of equivalence classes of objects $Ob(\mathcal{O})$ where addition is the direct sum and multiplication is defined by the decomposition (fusion) rules of the tensor product. We denote the ring by $Gr(\mathcal{O})$. For any premodular category $\mathcal{O}$ the adjoint action of the ring $Gr(\mathcal{O})$ on itself gives us the *incidence matrices*. We now restrict our attention to the category $\mathcal{F}$, where the Grothendieck semiring does not depend on the specific choice of a primitive $2l$th root of unity.

Definition 3.5. Fix $\lambda \in C_l$ and let $V_\lambda \otimes V_\mu = \bigoplus_{\nu \in C_l} N_{\lambda\mu}^{\nu} V_\nu$ be the decomposition into simple objects in $\mathcal{F}$. Then the *incidence matrix* corresponding to λ is

$$M_\lambda = (N_{\lambda\mu}^{\nu})_{\nu,\mu \in C_l}$$

We observed that the fusion coefficients are symmetric in all three variables in Section 2.4, so the matrices M_λ are each symmetric. We also observe that

$$\mathrm{Hom}_{\mathcal{F}}(V_\lambda, (V_{\mu_i} \otimes V_{\mu_j}) \otimes V_{\mu_k}) \cong \mathrm{Hom}_{\mathcal{F}}(V_\lambda, (V_{\mu_j} \otimes V_{\mu_i}) \otimes V_{\mu_k})$$

by the associativity and commutivity of the tensor product, so $(M_{\mu_i} M_{\mu_j})_{\lambda\mu_k} = (M_{\mu_j} M_{\mu_i})_{\lambda\mu_k}$ and the M_{μ_i} commute. Thus the set of incidence matrices $\mathcal{M} = \{M_\lambda : \lambda \in C_l\}$ is a commutative set of diagonalizable matrices, and hence simultaneously diagonalizable.

3.4 q-Characters

Unfortunately, empirical data indicates that the categorical q-dimension $\dim_q$ fails to be strictly positive on C_l for any choice of an lth root of unity q^2. More specifically, we used Mathematica to compute the values of $\dim_q$ for all lth roots of unity for several l and up to rank 3, and found that some object always had a negative $\dim_q$ value. So we need to generalize the notion of a q-dimension to have any hope of positivity. To do this we will have to divorce the Grothendieck semiring from the category itself, since the q-dimension is determined by a choice of q, the braiding, the duality and the twist. At the very minimum an appropriate generalization of the q-dimension must respect the tensor product in $\mathcal{F}$, that is, it must be a character of the fusion ring of $\mathcal{F}$.

Definition 3.6. A *character* for the fusion ring $Gr(\mathcal{F})$ is any function $f : C_l \to \mathbb{C}$ that satisfies

$$f(\lambda)f(\mu) = \sum_{\nu} N_{\lambda\mu}^{\nu} f(\nu) \tag{3.8}$$

where the $N_{\lambda\mu}^{\nu}$ are the fusion coefficients from Section 2.4.

Our main source of characters are the *q-characters* of $Gr(\mathcal{T})$ denoted χ_λ for any $\lambda \in P_+$ and $\nu \in Q$ defined as follows (recall the defintion of H_ν from Subsection 2.1.1):

$$\chi_\lambda(H_\nu) = \frac{1}{\delta_B(H_\nu)} \sum_{w \in W} \varepsilon(w) q^{\langle w(\lambda+\rho), \nu \rangle}$$

where

$$\delta_B(H_\nu) = \sum_{w \in W} \varepsilon(w) q^{\langle w(\rho), \nu \rangle}$$

is the *Weyl denominator*. Recall that $[n](q - q^{-1}) = q^n - q^{-n}$. An important computation due to Weyl [Wy] gives us the product form $\delta_B(H_\nu) = \prod_{\alpha \in \Phi_+} [\frac{1}{2}\langle \alpha, \nu \rangle]$ (see [GWa] Chapter 7 for a more modern treatment). The $\frac{1}{2}$ appears here because we have normalized the form $\langle\ ,\ \rangle$ to be twice the form used in the classical theory. But we needn't concern ourselves; $\frac{1}{2}\langle \alpha, \nu \rangle$ is a integer since both α and ν have integer entries.

Now let q be a fixed $2l$th root of unity, l odd. Notice that $\chi_\lambda(H_{2(\mu+\rho)}) = \tilde{S}_{\lambda\mu}$ so the normalized columns of the S-matrix are q-characters. We will also need the following more general q-characters:

Definition 3.7. Let $\mu \in P_+ \setminus \mathbb{Z}^k$ so that $\mu + \rho \in P_+ \cap Q$ (so $p(\mu + \rho) = 1$). Then for all $\lambda \in P_+$ we define

$$\dim_q^\mu(V_\lambda) = \chi_\lambda(H_{\mu+\rho}).$$

Notice that setting $\mu = 2\kappa+\rho$ in the above formula gives us $\dim_q^{2\kappa+\rho}(V_\lambda) = \tilde{S}_{\lambda\kappa}$ for any $\kappa \in C_l$ so that $\{\dim_q^\mu : \mu \in P_+\}$ includes the normalized columns of the S-matrix.

The importance of the q-characters is that for a fixed H_ν and q generic (that is, over the category $\mathcal{T}$) they satisfy the Property 2.1 and 2.2 mentioned in Section 2.4, and hence are characters of $Gr(\mathcal{T})$. In this setting these properties become:

1. $\chi_{\mathbf{0}}(H_\nu) = 1$
2. $\chi_\lambda(H_\nu)\chi_\mu(H_\nu) = \sum_i m_{\lambda\mu}^{\mu_i}\chi_{\mu_i}(H_\nu)$ where $V_\lambda \otimes V_\mu = \bigoplus_i m_{\lambda\mu}^{\mu_i} V_{\mu_i}$ as $U_q\mathfrak{so}_{2k+1}$-modules.

The first property is easily verified from the definitions, whereas the second is fundamental in the classical character theory.

Lemma 3.8. *For the specialization of a character $\chi_\kappa(H_\nu)$ of $Gr(\mathcal{T})$ to a $2l$ root of unity to be a character of $Gr(\mathcal{F})$ (i.e. satisfying equation 3.8) it is sufficient that:*

3. $\chi_\kappa(H_\nu) = \varepsilon(w)\chi_{w\cdot\kappa}(H_\nu)$ for all $\kappa \in C_l$, all $w \in W_l$ such that $w\cdot\kappa \in P_+$ and q a $2l$th root of unity, l odd.

Proof. Setting $W_\kappa = \{w \in W_l : w\cdot\kappa \in P_+\}$ for $\kappa \in \overline{C_l}$, Property 2 above becomes:

$$\chi_\lambda(H_\nu)\chi_\mu(H_\nu) = \sum_i m_{\lambda\mu}^{\mu_i}\chi_{\mu_i}(H_\nu) =$$
$$\sum_{\kappa\in\overline{C_l}}\left(\sum_{w\in W_\kappa}\varepsilon(w)m_{\lambda\mu}^{w\cdot\kappa}\right)\chi_\kappa(H_\nu) = \sum_{\kappa\in\overline{C_l}} N_{\lambda\mu}^\kappa\chi_\kappa(H_\nu)$$

since to every $\mu_i \in P_+$ there is a unique $\kappa \in D_0$ so that $w\cdot\kappa = \mu_i$ for some $w \in W_l$ and $N_{\lambda\mu}^\kappa = 0$ if $\kappa \in D_0 \setminus C_l$ (see Section 2.4). □

To prove Property 3 in the above lemma we need only verify it for simple reflections s_i, t_l since they generate W_l, and for the numerator of $\chi_\kappa(H_\nu)$ as the denominator $\delta_B(H_\nu)$ does not depend on κ. So the veracity of Property 3 will follow from the following lemma:

Lemma 3.9. $\sum_{w\in W}\varepsilon(w)q^{\langle w(r\cdot\kappa+\rho),\nu\rangle} = \varepsilon(r)\sum_{w\in W}\varepsilon(w)q^{\langle w(\kappa+\rho),\nu\rangle}$ *for r a simple reflection and $\nu\in Q$.*

Proof. Define $w'\in W$ by $w'(\lambda)=\lambda-\langle\lambda,\varepsilon_1\rangle\varepsilon_1$ and observe that $\varepsilon(w')=-1$ as w' just changes the sign of the first coordinate of λ. We compute:

$$\langle w(t_l\cdot\kappa+\rho),\nu\rangle = \langle t_l(\kappa+\rho)-\rho+\rho, w^{-1}(\nu)\rangle =$$
$$\langle(\kappa+\rho)-\langle\kappa+\rho,\varepsilon_1\rangle\varepsilon_1+l\varepsilon_1, w^{-1}(\nu)\rangle = \langle ww'(\kappa+\rho),\nu\rangle + l\langle\varepsilon_1,\nu\rangle$$

Since $l\langle\varepsilon_1,\nu\rangle$ is an even multiple of l and $\varepsilon(t_l)=-1$, we have:

$$\sum_{w\in W}\varepsilon(w)q^{\langle w(t_l\cdot\kappa+\rho),\nu\rangle} = \varepsilon(t_l)\sum_{w\in W}\varepsilon(w)q^{\langle w(\kappa+\rho),\nu\rangle}$$

after reindexing the sum. The computation for s_i is slightly less complicated, and just follows from the fact that $\chi_\kappa(H_\nu)$ is an antisymmetrization with respect to the Weyl group of the characters of the finite abelian group lP/Q. It can also be computed directly as for t_l. Thus we have proved the lemma. $\square$

Notice that this lemma implies that $\chi_\kappa(H_\nu)$ vanishes on $\mathcal{H}_l$, and hence on the tensor ideal $\mathcal{I}$. Thus the q-characters $\chi_\kappa(H_\nu)$ are indeed characters of the fusion ring $Gr(\mathcal{F})$.

Next we prove the following crucial:

Lemma 3.10. $\dim_q^{\Lambda_k}(V_\lambda)$ *is positive for all $\lambda\in C_l$ for $q=e^{\pi i/l}$.*

Proof. First we consider the numerator

$$\sum_{w\in W}\varepsilon(w)q^{\langle w(\lambda+\rho),\Lambda_k+\rho\rangle}$$

of $\dim_q^{\Lambda_k}(V_\lambda)$. Observe that the positive coroots $\check{\alpha}\in\check{\Phi}_+$ defined in Section 2.1 are $\frac{1}{2}$ the positive roots Φ_+^C of type C (corresponding to $\mathfrak{sp}_{2k}$). In the classical theory we would get exactly the positive roots of type C, but we are using twice the classical form. Furthermore $\Lambda_k+\rho=\rho'$ is one-half the sum of the positive roots of type C and is thus the sum of the positive coroots as we have defined them. Moreover, the Weyl group W is the same for these two algebras. Let (,) be the usual inner product on euclidian space, so that $2(a,b) = \langle a,b\rangle$. We have that

$$\sum_{w\in W}\varepsilon(w)q^{\langle w(\lambda+\rho),\Lambda_k+\rho\rangle} = \sum_{w\in W}\varepsilon(w)q^{\langle \lambda+\rho,w(\rho')\rangle} = \sum_{w\in W}\varepsilon(w)q^{(2(\lambda+\rho),w(\rho'))}$$
$$= \prod_{\beta\in\Phi_+^C}[(\lambda+\rho,\beta)] = \prod_{\check{\alpha}\in\check{\Phi}_+}[2(\lambda+\rho,\check{\alpha})] = \prod_{\check{\alpha}\in\check{\Phi}_+}[\langle\lambda+\rho,\check{\alpha}\rangle]$$

by the observations above and the classical Weyl denominator factorization for type C. The same computation for $\lambda = \mathbf{0}$ shows that the denominator of $\dim_q^{\Lambda_k}(V_\lambda)$ also factors nicely so that when we evaluate at $q = e^{\pi i/l}$ we get:

$$\dim_q^{\Lambda_k}(V_\lambda) = \prod_{\check{\alpha}\in\check{\Phi}_+}\frac{[\langle\lambda+\rho,\check{\alpha}\rangle]}{[\langle\rho,\check{\alpha}\rangle]} = \prod_{\check{\alpha}\in\check{\Phi}_+}\frac{e^{\langle\lambda+\rho,\check{\alpha}\rangle}-e^{-\langle\lambda+\rho,\check{\alpha}\rangle}}{e^{\langle\rho,\check{\alpha}\rangle}-e^{-\langle\rho,\check{\alpha}\rangle}}$$
$$= \prod_{\check{\alpha}\in\check{\Phi}_+}\frac{\sin(\langle\lambda+\rho,\check{\alpha}\rangle\pi i/l)}{\sin(\langle\rho,\check{\alpha}\rangle\pi i/l)}.$$

By the same analysis in Section 2.4 we see that when $\lambda\in C_l$, $\langle\lambda+\rho,\check{\alpha}\rangle < l$ for all $\check{\alpha}\in\check{\Phi}_+$ so that each factor in the above product is positive. □

We end this section with an important uniqueness theorem which relies on the classical theorem of Perron and Frobenius found in [Ga]. Recall that a positive matrix is a matrix whose entries are all strictly positive.

Proposition 3.11 (Perron-Frobenius). *A positive matrix A always has a positive real eigenvalue of multiplicity one whose modulus exceeds the moduli of all other eigenvalues. Furthermore the corresponding eigenvector may be chosen to have only positive real entries and is the unique eigenvector with that property.*

We now proceed to prove:

Theorem 3.12. *Evaluating $\dim_q^{\Lambda_k}(V_\lambda)$ at $e^{\pi i/l}$ gives the only q-character of $Gr(\mathcal{F})$ that is positive for all $\lambda\in C_l$.*

Proof. The key observation here is that for any function $f : C_l \to \mathbb{C}$ satisfying equation 3.8 the vector $\mathbf{f} = (f(\lambda))_{\lambda\in C_l}$ must be a simultaneous eigenvector of the set $\mathcal{M}$. In fact, using the definition of M_λ one computes that $M_\lambda(\mathbf{f}) = f(\lambda)\mathbf{f}$. So if we can show that $M_{\Lambda_k} \in \mathcal{M}$ has only one positive eigenvector we will have proved the lemma. Observe that for some odd integer s, the matrix $M_{\Lambda_k}^s + M_{\Lambda_k}^{s+1}$ has all positive entries. That is, every integral weight Weyl module appears in $V_{\Lambda_k}^{\otimes s} \otimes V_\lambda$ for λ a half-integral weight, and every half-integral weight module appears in $V_{\Lambda_k}^{\otimes s+1} \otimes V_\mu$ for μ an integral weight. (See the remark after Example 2.5). So one may apply the Perron-Frobenius Theorem to the matrix $M_{\Lambda_k}^s + M_{\Lambda_k}^{s+1}$ to see that it has a *unique* positive eigenvector. But M_{Λ_k} is a (symmetric) diagonalizable matrix, so it has the same eigenvectors as $M_{\Lambda_k}^s + M_{\Lambda_k}^{s+1}$. Since $\dim_q^{\Lambda_k}(V_\lambda)$ at $e^{\pi i/l}$ was shown to be positive in Lemma 3.10, we are done. $\square$

3.5 The Involution

Next we define an involution ϕ of C_l that will be central to the analysis of the q-characters of $\mathcal{F}$. Let $\gamma \in C_l$ be such that $|\gamma|$ is maximal, explicitly, $\gamma = (\frac{l-2k}{2}, \ldots, \frac{l-2k}{2})$. Further denote by w_1 the element of the Weyl group W such that $w_1(\mu_1, \ldots, \mu_k) = (\mu_k, \ldots, \mu_1)$. Define $\phi(\lambda) := \gamma - w_1(\lambda)$. It is clear that ϕ is a bijective map from C_l to itself and that $\phi^2(\lambda) = \lambda$, and that $\phi \notin W_l$ as no $\lambda \in P_+$ is fixed by ϕ. The following lemma describes the key property of ϕ.

Lemma 3.13. *For any q a primitive $2l$th root of unity the involution ϕ preserves $\dim_q^\mu$ (for $\mu \in P_+ \setminus \mathbb{Z}^k$) up to a sign, that is*

$$\dim_q^\mu(V_\lambda) = \pm \dim_q^\mu(V_{\phi(\lambda)}) \tag{3.9}$$

In particular (by setting $\mu = \rho$) this holds for the categorical q-dimension $\dim_q$ *of $\mathcal{F}$.*

Proof. Fix $\mu \in P_+ \setminus \mathbb{Z}^k$ and a choice of a primitive $2l$th root of unity q. First consider $\sum_{w\in W}\varepsilon(w)q^{\langle\lambda+\rho,w(\mu+\rho)\rangle}$ the numerator of $\dim_q^\mu(V_\lambda)$. We compute

$$\begin{aligned}\langle\phi(\lambda)+\rho,w(\mu+\rho)\rangle &= \langle\gamma-w_1(\lambda)+\rho,w(\mu+\rho)\rangle\\ &= \langle w_1(\gamma-\lambda+\rho+w_1(\rho)-\rho),w(\mu+\rho)\rangle\\ &= \langle\gamma+\rho+w_1(\rho),w_1w(\mu+\rho)\rangle+\langle\lambda+\rho,-w_1w(\mu+\rho)\rangle\\ &= l\cdot\sum_i(w_1w(\mu+\rho))_i+\langle\lambda+\rho,-w_1w(\mu+\rho)\rangle.\end{aligned}$$

Now $t(\mu) := \sum_i(w_1w(\mu+\rho))_i = \sum_i(w(\mu+\rho))_i$ is an integer whose parity is the same as that of $\sum_i(\mu+\rho)_i$ and depends only on μ (and the rank k), and $q^l=-1$ so $q^{l\cdot t(\mu)}=\pm1$ and we have

$$\begin{aligned}&\sum_{w\in W}\varepsilon(w)q^{\langle\phi(\lambda)+\rho,w(\mu+\rho)\rangle}=\sum_{w\in W}\pm\varepsilon(w)q^{\langle\lambda+\rho,-w_1w(\mu+\rho)\rangle}\\ &=\pm\sum_{w'\in W}\varepsilon(w')q^{\langle\lambda+\rho,w'(\mu+\rho)\rangle}\end{aligned}$$

where $w'=-w_1w$. Since the denominator of $\dim_q^\mu(V_\lambda)$ is independent of λ the lemma is true for $\mu\in P_+\cap\frac{1}{2}\mathbb{Z}^k\setminus\mathbb{Z}^k$. To see that it is true for any normalized column of the S-matrix, just replace μ by $2\kappa+\rho$ with $\kappa\in C_l$. □

Let us pause for a moment to nail down exactly which sign $\dim_q^{2\kappa+\rho}(V_{\phi(\lambda)})$ has in terms of $\dim_q^{2\kappa+\rho}(V_\lambda)$. We will use this later in analyzing the S-matrix. Here there are two factors governing signs of the q-characters: $\varepsilon(-w_1)$ and the parity of $\sum_i w(2\kappa+\rho)_i$. One has that:

$$\varepsilon(-w_1)=\begin{cases}(-1)^{k/2} & \text{for } k \text{ even}\\ (-1)^{(k-1)/2} & \text{for } k \text{ odd}\end{cases}$$

Furthermore (recalling the definition of $p(\kappa)$ from Section 2.2 we compute:

$$q^{l\sum_i w(2\kappa+\rho)_i}=\begin{cases}(-1)^k & \text{if } p(\kappa)=1\\ 1 & \text{if } p(\kappa)=-1\end{cases}$$

so we have the following result:

Scholium 3.14.

$$\dim_q^{2\kappa+\rho}(V_{\phi(\lambda)}) = \begin{cases} \dim_q^{2\kappa+\rho}(V_\lambda) & k \equiv 0 \bmod 4 \\ p(\kappa)\dim_q^{2\kappa+\rho}(V_\lambda) & k \equiv 1 \bmod 4 \\ -\dim_q^{2\kappa+\rho}(V_\lambda) & k \equiv 2 \bmod 4 \\ -p(\kappa)\dim_q^{2\kappa+\rho}(V_\lambda) & k \equiv 3 \bmod 4 \end{cases}$$

From this we can easily see that the S-matrix for $\mathcal{F}$ is never invertible–the rows corresponding to λ and $\phi(\lambda)$ always differ by ± 1. This result will be used in Section 6.2.

The following important lemma gives the decomposition rule for tensoring with the object in $\mathcal{F}$ labelled by γ.

Lemma 3.15. *$V_\gamma \otimes V_\mu = V_{\phi(\mu)}$ for all $\mu \in C_l$.*

Proof. By Lemmas 3.13 and 3.10 we know that $\dim_q^{\Lambda_k}(V_\gamma) = \dim_q^{\Lambda_k}(V_0) = 1$ since $\phi(\mathbf{0}) = \gamma$. So

$$\dim_q^{\Lambda_k}(V_\gamma \otimes V_\mu) = \dim_q^{\Lambda_k}(V_\mu) = \dim_q^{\Lambda_k}(V_{\phi(\mu)}).$$

Recall from 2.4 that the simple U-module decomposition of $V_\gamma \otimes V_\mu$ into simple objects is $\bigoplus_\nu N_{\gamma\mu}^\nu V_\nu$ with

$$N_{\gamma\mu}^\nu = \sum_{W_\nu} \varepsilon(w) m_{\gamma\mu}^{w\cdot\nu}$$

where $W_\nu = \{w \in W_l : w \cdot \nu \in P_+\}$ and $m_{\gamma\mu}^{w\cdot\nu}$ is the multiplicity of $V_{w\cdot\nu}$ in the decomposition of $V_\gamma \otimes V_\mu$ as $U\mathfrak{so}_{2k+1}$-modules. Observe that the weight $\phi(\mu) = \gamma - w_1(\mu)$ is in C_l and $m_{\gamma\mu}^{\phi(\mu)} = 1$ (see Section 2.3). The only way that $V_\phi(\mu)$ might not appear in the $\mathcal{F}$ decomposition is if $\phi(\mu)$ were equal to a reflection (under the dot action of W_l) of $\gamma + \kappa$ for some $\kappa \in P(\mu)$ (notice this also covers weights in other Weyl chambers). To see that this is impossible, we use a geometric argument, although it is really nothing more than an adaptation of the classical outer multiplicity formula. First note that γ is a positive distance from all walls of reflection under the dot action of W_l. Next observe that the straight line segment from γ to $\gamma+\kappa$ has Euclidian length $|\kappa| \leq |\mu|$. So the reflected piecewise linear path from γ to $w \cdot (\gamma+\kappa)$ will not be straight, and will have total length $|\kappa|$ as well. Thus the straight line segment from γ to $w \cdot (\gamma+\kappa)$ must have length strictly less than $|\mu|$, whereas the straight line segment from γ to $\phi(\mu)$ has length $|\mu|$. So $V_\phi(\mu)$ is a U-submodule of $V_\gamma \otimes V_\mu$. But since $\dim_q^{\Lambda_k}$ is positive on C_l and

$$\dim_q^{\Lambda_k}(V_{\phi(\mu)}) = \dim_q^{\Lambda_k}(V_\gamma \otimes V_\mu) = \sum_\nu N_{\gamma\mu}^\nu \dim_q^{\Lambda_k}(V_\nu)$$

it is clear that $V_{\phi(\mu)}$ is the *only* submodule that appears in the decomposition. $\square$

Chapter 4

Categories from BMW-Algebras

In this section we will discuss the BMW-algebras $C_f(r,q)$ and the semi-simple quotients we are interested in. These algebras are quotients of the group algebra of Artin's braid group $\mathcal{B}_f$ and were studied extensively in [W1] and [TW2], and more recently in [TuW2].

Definition 4.1. Let $r, q \in \mathbb{C}$ and $f \in \mathbb{N}$, then $C_f(r,q)$ is the $\mathbb{C}$-algebra with invertible generators $g_1, g_2, \ldots, g_{f-1}$ and relations:

B1 $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$,

B2 $g_i g_j = g_j g_i$ if $|i-j| \geq 2$,

R1 $e_i g_i = r^{-1} e_i$,

R2 $e_i g_{i-1}^{\pm 1} e_i = r^{\pm 1} e_i$,

where e_i is defined by

E1 $(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}$

Notice that $(E1)$ and $(R1)$ imply

$$(g_i - r^{-1})(g_i - q)(g_i + q^{-1}) = 0$$

for all i. So the image of g_i on any finite dimensional representation has minimal polynomial with distinct roots provided $q^2 \neq 1$ and $r \notin \{q, -q^{-1}\}$. Notice further that the image of e_i is a multiple of the projection onto the g_i-eigenspace corresponding to eigenvalue r^{-1}. There exists a trace tr on the family of algebras $C_f(r,q)$ uniquely determined by the values on the generators, and inductively defined by the Markov property (see [W1]). Explicitly we have $tr(1) = 1$, $tr(g_i) = r(\frac{q-q^{-1}}{r-r^{-1}+q-q^{-1}})$, and $tr(axb) = tr(ab)tr(x)$ for $a, b \in C_{f-1}(r,q)$ and $x \in \{g_{f-1}, e_{f-1}, 1\}$. The existence of such a trace comes from the well-known Kauffmann link invariant. Observe that $C_f(r,q)$ has a number of automorphisms, for example replacing the generators by their negatives or inverses does not change the structure. These automorphisms induce isomorphisms between BMW-algebras with different parameters. For example $C_f(r,q) \cong C_f(-r,-q)$. This freedom has been completely analyzed in [TuW2], and will be mentioned later.

4.1 The Relevant Specialization

Now assume q is a primitive $2l$th root of unity, l odd and $r = -q^{2k}$. Most of what follows is true of any choice of a root of unity q and r a power of q, but we restrict our attention to the present case for brevity's sake. Let $\mathcal{A}_f$ be the annihilator ideal of tr on $C_f(-q^{2k}, q)$. Then the algebras $E_f(k,l) := C_f(-q^{2k}, q)/\mathcal{A}_f$ are semisimple and finite dimensional and are thus isomorphic with a direct sum of full matrix algebras. Henceforth we will denote these algebras simply by E_f, and for the images of the $g_i^{\pm 1}$ in this quotient we will denote by the same symbol for ease of notation. By semisimplicity we can decompose E_f via minimal idempotents.

Definition 4.2. An *idempotent* in an algebra is an element x so that $x^2 = x$. Such an idempotent is called minimal if it is nonzero and if whenever $x = w + z$ with w and z idempotents then w or z is zero. We call two idempotents x and y orthogonal if $xy = yx = 0$.

Notice that $E_f \subset E_{f+s}$, $s \geq 0$ since $C_f(r,q) \subset C_{f+s}(r,q)$ and the tr function and its annihilator are defined inductively to have the Markov property and are thus compatible with the embeddings. We define E_∞ to be the inductive limit of the E_f as $f \to \infty$. Two idempotents $x \in E_{f_1}$ and $y \in E_{f_2}$ are equivalent in E_∞ if there are elements u, v in some E_f so that $x = uv$ and $y = vu$. Another definition is that two idempotents x, y are equivalent precisely if $xE_f \cong yE_f$ as right E_f-modules.

Define a category $\mathcal{V}$ whose objects are idempotents in E_∞, and define the morphisms spaces as in [TW2] Section 7.2. We will not include this as it is somewhat involved and we are mostly only concerned with the Grothendieck semiring. The tensor product on $\mathcal{V}$ comes from the embeddings of $E_n \subset E_{n+m}$. To understand this product on the semiring level we use the method of Goodman and Wenzl in [GoW], whereas at the categorical level the process is essentially due to Turaev.

Definition 4.3. A representation (π, A) of the inductive limit $\mathcal{B}_\infty$ of $\mathcal{B}_n$ into the invertible elements of an algebra A is called approximately finite if $A_n := \pi(\mathbb{C}\mathcal{B}_n)$ is semisimple and finite dimensional for each n.

For any n, m there is a group homomorphism $\mathrm{shift}_m : \mathcal{B}_n \to \mathcal{B}_{m+n}$ defined on generators by $\sigma_i \to \sigma_{i+m}$. In fact there is an element $\sigma_{m,n} \in \mathcal{B}_{m+n}$ (see [GoW]) so that

1. $\sigma_{m,n}^{-1}\sigma_i\sigma_{m,n} = \sigma_{i+m}$ for $1 \leq i \leq n-1$
2. $\sigma_{m,n}^{-1}\sigma_{n+j}\sigma_{m,n} = \sigma_j$ for $1 \leq j \leq m-1$

If (π, A) is an approximately finite representation of $\mathcal{B}_\infty$, π induces a homomorphism $\mathrm{shift}_m : A_n \to A_{m+n}$. If we denote by $[x]$ the equivalence class of the idempotent $x \in A_m$ and $[y]$ for $y \in A_n$ we can define a product by: $[x][y] = [x \cdot \mathrm{shift}_m(y)]$. Here we identify $x \in A_m \subset A_{m+n}$ with its image under the inclusion. (This comes from the inclusion $\mathcal{B}_m \subset \mathcal{B}_{m+n}$ and the fact that the trace is inductively defined.)

Proposition 4.4 (Goodman-Wenzl). *The product $\otimes$ above is:*
well-defined, associative and commutative and respects orthogonality: if $x = x' +$

x'' with x' and x'' orthogonal idempotents then $x'\mathrm{shift}_m(y)$ and $x''\mathrm{shift}_m(y)$ are orthogonal idempotents.

We may now apply this to E_∞ to describe the tensor product on $\mathcal{V}$ (again, just at the level of the Grothendieck semiring). It can be shown (by relating the BMW-algebras to Hecke algebras, see [W1]) that the equivalence classes of minimal idempotents in E_∞ are labelled by Ferrers diagrams. In fact, the equivalence classes of minimal idempotents in E_∞ are finite in number–the labelling set is given below. So we have the following for minimal idempotents $p_\lambda, p_\mu \in E_\infty$:

$$[p_\lambda][p_\mu] = \sum_\nu c^\nu_{\lambda\mu}[p_\nu] \tag{4.1}$$

where $c^\nu_{\lambda\mu}$ is the number of minimal idempotents equivalent to p_ν that appear in the decomposition of $p_\lambda\mathrm{shift}_m(p_\mu)$ into a sum of minimal idempotents. Next we give the exact definition of the labelling set for (classes of) inequivalent minimal idempotents.

Definition 4.5. Let λ_s be the number of boxes in the sth row of the Ferrers diagram λ and λ'_s be the number of boxes in the sth column. The simple objects of $\mathcal{V}$ are labelled by the following Ferrers diagrams:

$$\Gamma(k,l) = \{\lambda : \lambda'_1 + \lambda'_2 \leq 2k+1, \lambda_1 \leq (l-2k-1)/2\}$$

We will call the elements of $\Gamma(k,l)$ weights, and we now denote the simple objects of $\mathcal{V}$ by X_λ. By semisimplicity of E_f, any object in $\mathcal{V}$ can be expressed uniquely as a sum of X_λ. One computes $|\Gamma(k,l)| = 2\binom{\frac{l-1}{2}}{k}$ so $|\Gamma(k,l)| = |C_l|$. The Markov trace on E_∞ gives rise to a q-dimension on the simple objects $\mathcal{V}$ by evaluating the trace on minimal idempotents. It is related to Kauffmann's link invariant [Kf] and can be computed by evaluating characters of the group $O(2k+1)$. This q-dimension $Q_\lambda(q)$ is defined for any Ferrers diagram as follows (see [W1]). By the (i,j)th box in a Ferrers diagram we mean the box in the ith row and jth column. For each box in a fixed diagram λ we define the two quantities $d(i,j)$ and

$h(i,j)$ by:

$$d(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & \text{if } i \leq j \\ -\lambda'_i - \lambda'_j + i + j - 1 & \text{if } i > j \end{cases}$$

and $h(i,j) = \lambda_i - i + \lambda_j{}' - j + 1$ (the hook length). With these functions in hand we can now define the q-dimension for $\mathcal{V}$.

Definition 4.6. For each λ let

$$Q_\lambda(q) = \prod_{(j,j)\in\lambda} \frac{[-2k - (\lambda_j - \lambda'_j)] + [h(j,j)]}{[h(j,j)]} \prod_{(i,j)\in\lambda, i\neq j} \frac{[-2k - d(i,j)]}{[h(i,j)]}$$

where $[n]$ is the q-number.

By semisimplicity one can extend this function linearly to any sum of simple objects, and the character theory of $O(2k+1)$ shows that this is indeed a q-dimension. The formula for more general BMW-algebras and the complete derivation can be found in [W1]. However, note that in that paper the equations are slightly different due to the fact that the transposed labelling set is used. We see that we have the right $Q_\lambda(q)$ function by computing that if $\lambda'_1 + \lambda'_2 = 2k+2$ (respectively if $\lambda_1 = (l-2k+1)/2$) that the factor corresponding to the $(2,1)$ box (respectively the $(1,1)$ box) is zero. This point should be emphasized: the BMW-algebras $C_n(r,q)$ and $C_n(r,-q^{-1})$ are isomorphic, with the isomorphism being achieved by transposing the labels of the minimal idempotents (see [W1]). Notice that if $q^l = -1$ with l odd then $(-q^{-1})^l = 1$ and so we may assume with out loss of generality that q is a primitive $2l$th root of unity to cover all cases where q^2 is a primitive lth root of unity. We alluded to this in the beginning of Section 2.4, and by the end of chapter 5 it will be clear that our tacit assumptions about q do not exclude any cases we have claimed to have covered.

The semiring of $\mathcal{V}$ can also be described as follows. Let $Rep(O(2k+1))$ be the tensor category of representations of the Lie group $O(2k+1)$. Let $\mathcal{J}$ be the ideal in $Gr(Rep(O(2k+1))$ generated by the simple objects W_λ with $Q_\lambda(q) = 0$. The analysis of E_∞ in [W1] shows that $Gr(\mathcal{V}) \cong Gr(Rep(O(2k+1))/\mathcal{J}$. An explicit

Littlewood-Richardson rule has not been derived for the category $\mathcal{V}$, but we can decompose a certain class of tensor products as follows:
Let $\lambda, \mu \in \Gamma(k,l)$, and W_λ, W_μ be the corresponding irreducible $O(2k+1)$-modules. If for every $W_\nu \subset W_\lambda \otimes W_\mu$ either $\nu \in \Gamma(k,l)$ or $Q_\nu(q) = 0$ then $X_\lambda \otimes X_\mu$ decomposes as the direct sum of all X_ν with $\nu \in \Gamma(k,l)$ with the same multiplicities as in the $O(2k+1)$-module decomposition.

Example 4.7. The object $X = X_{[1]}$ generates the category in the sense that every simple object in $\mathcal{V}$ appears as a direct summand of $X^{\otimes n}$ for some n. It is easy to describe the explicit decomposition algorithm for $X \otimes X_\lambda$ with $\lambda \in \Gamma(k,l)$. The corresponding representation $W_{[1]}$ of $O(2k+1)$ is the vector representation for which the tensor product rules are quite simple. As $O(2k+1)$-modules the decomposition is $W_{[1]} \otimes W_\lambda = \bigoplus_\nu W_\nu$ where ν is a Ferrers diagram gotten from λ by adding or subtracting one box. Thus the decomposition of $X \otimes X_\lambda$ is the same after discarding all $\nu \notin \Gamma(k,l)$. For example,

$$X^{\otimes 2} = \mathbb{1} \oplus X_{[2]} \oplus X_{[1^2]}$$

where $\mathbb{1}$ is the identity object corresponding to $1 \in E_0 = \mathbb{C}$.

The following key proposition is in [TuW2] (Theorem 8.5), and shows that the morphisms spaces in $\mathcal{V}$ are precisely the algebras from which the category is derived.

Proposition 4.8. *For each n we have $\Xi : E_n \cong \mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ as $\mathbb{C}$-algebras. Moreover, Ξ respects embeddings:*
if $x \in E_n$ and $y \in E_m$ are idempotents, then the idempotents

$$(\Xi(x) \otimes \mathbf{1}_X^{\otimes m}) \circ (\mathbf{1}_X^{\otimes n} \otimes \Xi(y))$$

and

$$\Xi(x\mathit{shift}_n(y))$$

are equivalent in $\mathrm{End}_{\mathcal{V}}(X^{\otimes(m+n)})$.

This proposition shows that the centralizer algebras $\mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ are generated by the images of the g_i.

4.1.1 $\mathcal{V}$ Summarized

The following is a compendium of the key properties of the category $\mathcal{V}$. These can be found in the references at the beginning of this chapter.

1. $\mathcal{V}$ is a premodular category. The braiding comes from the images of the elements $\sigma_{m,n}$ in E_{m+n}. In particular, in $\mathrm{End}_{\mathcal{V}}(X^{\otimes 2})$ the braiding matrix is $C_{X,X} = \Xi(g_1)$. Other braidings are given by replacing $C_{X,X}$ by its inverse or negative (or both).

2. The eigenvalues of $C_{X,X}$ on $X^{\otimes 2}$ restricted to $\mathbb{1}$, $X_{[2]}$ and $X_{[1^2]}$ are $-q^{-2k}$, q and $-q^{-1}$ respectively.

3. $Q_{[1]}(q) = \frac{q^{-2k}-q^{2k}}{q-q^{-1}} + 1$ with the first braiding above. Other braidings only affect the sign of $Q_{[1]}(q)$.

4. The isomorphism Ξ respects embeddings.

5. $Gr(\mathcal{V}) \cong Gr(Rep(O(2k+1)))/\mathcal{J}$.

Chapter 5

The Equivalence

This chapter contains the main result: $\mathcal{F}$ and $\mathcal{V}$ are equivalent as tensor categories. It has recently been proved (see [TuW2] Theorem 9.5) that the category $\mathcal{V}$ is completely determined by:

1. The Grothendieck semiring $Gr(\mathcal{V})$,
2. The eigenvalues of the braiding morphism $C_{X,X}$.

This theorem implies that any braided tensor category $\mathcal{O}$ with $Gr(\mathcal{O}) \cong Gr(\mathcal{V})$ with the same eigenvalues for the braiding matrices is equivalent to $\mathcal{V}$. So we must show that there exists a q so that $\mathcal{F}$ has the same Grothendieck semiring as $\mathcal{V}$ and that there is an object $V \in \mathcal{F}$ so that the eigenvalues of one of $\pm\check{R}_{VV}^{\pm 1}$ match those of $C_{X,X}$. To achieve this we must first recall some vital facts–proofs of which may be found in previous chapters or in the references. The equivalence will be staged in several somewhat technical steps. The proof is outlined as follows:

I For any n the centralizer algebras $\mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ and $\mathrm{End}_{\mathcal{F}}(V^{\otimes n})$ are isomorphic.

II There exists a braiding on $\mathcal{F}$ such that the representation of the braid group on $V^{\otimes n}$ factors through E_n for all n, and the traces on each are compati-

Table 5.1: Tensor Categories

Category	Labelling Set	Objects
$Rep(O(2k+1))$	Diagrams λ, $\lambda_1' + \lambda_2' \leq 2k+1$	W_λ
$\mathcal{V}$	$\Gamma(k,l)$	X_λ
$Rep(U_q\mathfrak{so}_{2k+1})$, $\lvert q\rvert \neq 1$	P_+	V_λ
$\mathcal{F}$	C_l	V_λ

ble. Using part I we show that this map is an isomorphism that respects embeddings.

III Combining part II with the isomorphism Ξ in the previous chapter, we get a family of isomorphisms between $\mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ and $\mathrm{End}_{\mathcal{F}}(V^{\otimes n})$ that preserves the braiding morphisms. This shows that $\mathcal{F}$ and $\mathcal{V}$ are equivalent as abstract tensor categories. We then identify precisely the correspondence for simple objects in $\mathcal{F}$ and $\mathcal{V}$.

5.1 Related Tensor Categories

Several tensor categories will be bandied about in what follows. Recall first the following sets:

1. $\Gamma(k,l) = \{\lambda : \lambda_1' + \lambda_2' \leq 2k+1, \lambda_1 \leq (l-2k-1)/2\}$. Here λ is a Ferrer's diagram, and λ_i' is the number of boxes in the ith column.

2. $P_+ = \{\lambda \in \mathbb{Z}^k \cup (\mathbb{Z}^k + \frac{1}{2}(1,1,\ldots,1)) : \lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \geq 0\}$

3. $C_l = \{\lambda \in P_+ : \frac{l-2k}{2} \geq \lambda_1\}$.

Table 5.1 will serve as a lexicon of notation. The first column is the category, the second the labelling set for simple objects, and the third the notation used for the simple object labelled by λ.

Next we note a few homomorphisms that exist between the Grothendieck semirings of these tensor categories.

1. As we mentioned in the previous chapter, the ring $Gr(\mathcal{V})$ is a quotient of $Gr(Rep(O(2k+1)))$. Provided $\mu \in \Gamma(k,l)$ we have:

$$\dim \mathrm{Hom}_{\mathcal{V}}(X_\mu, X_{[1]} \otimes X_\lambda) = \dim \mathrm{Hom}_{O(2k+1)}(W_\mu, W_{[1]} \otimes W_\lambda).$$

2. From the classical theory we know that restricting $O(2k+1)$-modules to $SO(2k+1)$ and then taking the differential of the representation gives a homomorphism from $Gr(Rep(O(2k+1)))$ to $Gr(Rep(\mathfrak{so}_{2k+1}))$. Recall from Section 2.3 that the dominant weights of $O(2k+1)$ are integral (Young diagrams) whereas $\mathfrak{so}_{2k+1}$ has both integral and half-integral dominant weights, so this homomorphism is two-to-one. From this we deduce:

$$\dim \mathrm{Hom}_{\mathcal{V}}(W_\mu, W_{[1]} \otimes W_\lambda) = \dim \mathrm{Hom}_{\mathfrak{so}_{2k+1}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\overline{\lambda}})$$

where $\overline{\mu}$ and $\overline{\lambda}$ are the integral weights in P_+ defined in Subsection 2.3.2.

3. For generic q, $Gr(Rep(\mathfrak{so}_{2k+1}))$ and $Gr(Rep(U_q\mathfrak{so}_{2k+1}))$ are isomorphic. For this reason we denote the simple objects from both categories by V_λ.

4. The category $\mathcal{F}$ is obtained from $Rep(U_q\mathfrak{so}_{2k+1})$ as a quotient. The explicit Littlewood-Richardson rule was described in Proposition 2.4. Heedless of any potential confusion, we denote the simple objects in $\mathcal{F}$ by V_λ as well. Recall from example 2.6 that for any integral weight $\lambda \in C_l$:

$$V_\mu \subset V_{\Lambda_1} \otimes V_\lambda \iff V_\mu \subset V_{\Lambda_1} \otimes V_\lambda, \ \mu \in C_l.$$

5.2 Step One

We will write $V_{\phi(\Lambda_1)} = V$. We wish to show that

$$\mathrm{End}_{\mathcal{V}}(X^{\otimes n}) \cong \mathrm{End}_{\mathcal{F}}(V^{\otimes n})$$

as $\mathbb{C}$-algebras for all $n \geq 0$. Define a bijection $\Psi : \Gamma(k,l) \to C_l$ by

$$\Psi(\lambda) = \begin{cases} \bar{\lambda}, & \text{if } |\lambda| \text{ is even,} \\ \phi(\bar{\lambda}), & \text{if } |\lambda| \text{ is odd.} \end{cases} \tag{5.1}$$

Observing that the tensor product of any simple object in $\mathcal{V}$ (resp. $\mathcal{F}$) with the generating object X (resp. V) is multiplicity free, the algebras $\mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ and $\mathrm{End}_{\mathcal{F}}(V^{\otimes n})$ are isomorphic once we prove:

Lemma 5.1. *Let* $\mu, \lambda \in \Gamma(k,l)$. *Then*

$$\dim \mathrm{Hom}_{\mathcal{V}}(X_\mu, X \otimes X_\lambda) = \dim \mathrm{Hom}_{\mathcal{F}}(V_{\Psi(\mu)}, V \otimes V_{\Psi(\lambda)}).$$

Proof. Using the first homomorphism of Grothendieck semirings above and the assumption that $\mu \in \Gamma(k,l)$, we have

$$\dim \mathrm{Hom}_{\mathcal{V}}(X_\mu, X \otimes X_\lambda) = \dim \mathrm{Hom}_{O(2k+1)}(W_\mu, W_{[1]} \otimes W_\lambda).$$

Restricting to $SO(2k+1)$, differentiating and applying the third homomorphism above we have

$$\dim \mathrm{Hom}_{U_q\mathfrak{so}_{2k+1}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\overline{\lambda}}) = \dim \mathrm{Hom}_{O(2k+1)}(W_\mu, W_{[1]} \otimes W_\lambda).$$

Now we split into the two cases from the definition of Ψ:
Case I: $|\lambda|$ is even (so $|\mu|$ is odd)
Since $\overline{\mu} \in C_l$ and $V_{\Psi(\lambda)} = V_{\overline{\lambda}}$ we see that

$$\dim \mathrm{Hom}_{\mathcal{F}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\Psi(\lambda)}) = \dim \mathrm{Hom}_{U_q\mathfrak{so}_{2k+1}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\overline{\lambda}}) \tag{5.2}$$

Lemma 3.15 implies that $V_\gamma \otimes V_{\overline{\mu}} = V_{\phi(\overline{\mu})} = V_{\Psi(\mu)}$ as objects in $\mathcal{F}$, and similarly $V_\gamma \otimes V_{La_1} = V$. So tensoring with V_γ (see example 5.2) we have:

$$\dim \mathrm{Hom}_{\mathcal{F}}(V_{\Psi(\mu)}, V \otimes V_{\Psi(\lambda)}) = \dim \mathrm{Hom}_{\mathcal{F}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\Psi(\lambda)}).$$

Case II: $|\lambda|$ is odd (so $|\mu|$ is even)
In this case $V_{\Psi(\lambda)} = V_\gamma \otimes V_{\overline{\lambda}}$ and $V_{\Psi(\mu)} = V_{\overline{\mu}}$ so using the fact that $V_\gamma \otimes V_\gamma = \mathbb{1}$ we derive similarly that

$$\dim \mathrm{Hom}_{\mathcal{F}}(V_{\Psi(\mu)}, V \otimes V_{\Psi(\lambda)}) = \dim \mathrm{Hom}_{\mathcal{F}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\Psi(\lambda)})$$

in this case. $\square$

This lemma implies that

$$\dim \mathrm{Hom}_{\mathcal{V}}(X_\nu, X^{\otimes n}) = \dim \mathrm{Hom}_{\mathcal{F}}(V_{\Psi(\nu)}, V^{\otimes n})$$

by an easy induction argument. Since both categories are semisimple we conclude that there is a $\mathbb{C}$-algebra isomorphism:

$$F : \mathrm{End}_{\mathcal{V}}(X^{\otimes n}) \cong \mathrm{End}_{\mathcal{F}}(V^{\otimes n}). \tag{5.3}$$

Hence and further we abbreviate $V_{(2,0,\ldots,0)} = V_{\Psi([2])} = V_1$ and $V_{(1,1,0,\ldots,0)} = V_{\Psi([1^2])} = V_2$ for ease of notation.

5.3 Step Two

In this section we show that the algebras E_f and $\mathrm{End}_{\mathcal{F}}(V^{\otimes f})$ are isomorphic. Moreover, the isomorphism respects the embeddings

$$\mathrm{End}_{\mathcal{F}}(V^{\otimes f_1}) \otimes \mathrm{End}_{\mathcal{F}}(V^{\otimes f_2}) \subset \mathrm{End}_{\mathcal{F}}(V^{\otimes (f_1+f_2)})$$

given by $a \otimes b \to (a \otimes \mathbf{1}^{\otimes f_2}) \circ (\mathbf{1}^{\otimes f_1} \otimes b)$. We have a family of representations π of $\mathbb{C}\mathcal{B}_f$ on $\mathrm{End}_{\mathcal{F}}(V^{\otimes f})$ given by $\sigma_i \to R_i$ as in Section 2.5 corresponding to the braiding matrix $\check{R}_{VV}$, which we will denote by R in this section. Moreover, π respects embeddings by approximate finiteness, so it only remains to check that the representation of $\mathbb{C}\mathcal{B}_f$ on $V^{\otimes f}$ factors through E_f and the trace on E_f is compatible with the categorical trace on $\mathcal{F}$. So we must see that the operators R_i defined in Section 2.5 satisfy the defining relations and trace restrictions of $C_f(-q^{2k}, q)$ of chapter 4. In fact, it is sufficient to check the relations and traces for $f = 3$ since all the relations (except $B2$, which follows directly from the definitions) only involve generators whose indices differ by at most 1. The two main computations are:

(a) There is an action of $\mathbb{C}\mathcal{B}_2$ on $V^{\otimes 2}$ so that the image of σ_1 has eigenvalues $-q^{-2k}$, q and $-q^{-1}$ respectively on the submodules $\mathbb{1}$, V_1 and V_2 for some primitive $2l$th root of unity q.

(b)

$$\pm \dim_q(V) = \frac{q^{-2k} - q^{2k}}{q - q^{-1}} + 1$$

We do not fix a specific braiding so this gives us the freedom to rescale R by multiplying by any 4th root of unity (see [TuW1] Section 3.3) to verify the above. We begin with a fixed primitive $2l$th root of unity q.

From Proposition 2.7, we compute the eigenvalues of R_1^2 on $\mathbb{1}$, V_1 and V_2 to be $(-1)^k q^{-8k}$, $(-1)^k q^4$ and $(-1)^k q^{-4}$. Using Lemma 3.13 and the definition of $\dim_q$ from Section 2.4 we compute

$$\pm \dim_q(V) = \dim_q(V_{\Lambda_1}) = \frac{q^{4k} - q^{-4k}}{q^2 - q^{-2}} + 1.$$

By multiplying by an appropriately chosen 4th root of unity, we may eliminate the factor $(-1)^k$. Next notice that $\tilde{q} = -q^2$ is also a primitive $2l$th root of unity, so if we replace $-q^2$ by $\tilde{q}$ the eigenvalues of R^2 become $\tilde{q}^{-4k}$,$\tilde{q}^2$ and $\tilde{q}^{-2}$. So there are eight possible choices for the eigenvalues of the R, corresponding to the sign choices: $\pm\tilde{q}^{-2k}$, $\pm\tilde{q}^{-1}$ and $\pm\tilde{q}$. Replacing $-q^2$ by $\tilde{q}$ in the q-dimension formula above gives us:

$$\pm \dim_q(V) = \frac{\tilde{q}^{-2k} - \tilde{q}^{2k}}{\tilde{q} - \tilde{q}^{-1}} + 1. \tag{5.4}$$

So we have verified condition (b) above. The following result from [TuW2] (proof of Lemma 6.3) allows us to compute the q-dimension of the object V up to a sign from the eigenvalues of R on $V^{\otimes 2}$.

Proposition 5.2. *Let $\mathcal{O}$ be a ribbon category and X a simple object in $\mathcal{O}$ such that $X^{\otimes 2} = Y \oplus Z \oplus \mathbb{1}$ where $\mathbb{1}$ is the identity object and Y, Z are simple objects. If the eigenvalues of the image of the braid generators on $X^{\otimes 2}$ are $\{c_1, c_2, c_3\}$ respectively then the categorical q-dimension of the object X is:*

$$\pm \dim_q(X) = \frac{c_3^2 + c_1 c_2 - c_3(c_1 + c_2)}{c_3(c_1^{-1} + c_2^{-1})} \tag{5.5}$$

Applying this to the current situation with the category $\mathcal{F}$ and comparing with equation 5.4 we can eliminate all but two of the eight possible sign choices: $\tilde{q}^{-2k}$,

$\tilde{q}$ and $-\tilde{q}^{-1}$ and $-\tilde{q}^{-2k}$, $\tilde{q}$ and $\tilde{q}^{-1}$. But multiplying R by -1 changes the former into the latter, so we have verified condition (a) above. To give an idea of the computation involved consider the case where $c_1 = \tilde{q}$ and $c_2 = \tilde{q}^{-1}$. Let $a = \tilde{q}^{-2k}$ and $c_3 = \pm a$. From equation 5.4 and the proposition above we get:

$$\frac{\pm(a + a^{-1})}{\tilde{q} + \tilde{q}^{-1}} - 1 = \pm\frac{a - a^{-1}}{\tilde{q} - \tilde{q}^{-1}} + 1$$

Solving for a with the four possible sign choices gives us

$$a \in \{-\tilde{q}^{\pm 3}, \tilde{q}^{\pm 1}, -\tilde{q}^{\pm 1}\}$$

which contradicts $a = \tilde{q}^{-2k}$.

Let t denote the multiple of R resulting from the various changes we made, and t_i the corresponding multiple of R_i. Now we are in a position to show that the relations $B1, B2, R1$ and $R2$ are satisfied, as well as the compatibility of the trace. Define $e = 1 - \frac{t - t^{-1}}{\tilde{q} - \tilde{q}^{-1}}$ so that e_i is defined as in 4. We will need the following fact from [TuW2] (Lemma 5.2) which we will state, in the current situation:

Lemma 5.3. *Let $b \in \mathrm{End}_{\mathcal{F}}(V^{\otimes 2})$ and let p be the projection onto $\mathbb{1} \in V^{\otimes 2}$. Then*

$$(\mathbf{1}_V \otimes p) \circ (b \otimes \mathbf{1}_V) \circ (\mathbf{1}_V \otimes p) = tr(b)(\mathbf{1}_V \otimes p)$$

(This result follows from the duality in the category.) We have the following results which are in some cases obvious and in other cases tedious computations:

1. $B1$ and $B2$ are automatically satisfied by the t_i as they are not affected by the changes we made to the R_i.

2. $te = -\tilde{q}^{-2k}e$ as e is by its very definition a multiple of the projection onto the trivial representation $\mathbb{1} \subset V^{\otimes 2}$ on which t acts by $-\tilde{q}^{2k}$.

3. A direct computation using the categorical trace and the known q-dimensions yields $tr_{\mathcal{F}}(t) = -\tilde{q}^{2k}(\frac{\tilde{q} - \tilde{q}^{-1}}{\tilde{q}^{-2k} - \tilde{q}^{2k} + \tilde{q} - \tilde{q}^{-1}})$ as required. (See Proposition 5.2.)

4. That $tr_{\mathcal{F}}(e) = tr_{\mathcal{F}}(t)/(-\tilde{q}^{2k})$ follows from the defintion of e.

5. Lemma 5.3 shows that $e_2 t_1^{\pm 1} e_2 = r^{\pm 1} e_2$, since e is a multiple of the projection onto $\mathbb{1}$.

Thus the braid representation π factors through E_n for all n, and since the traces on each satisfy the Markov property they are compatible. We denote the resulting homomorphism by Ω.

Lemma 5.4. *The map $\Omega : E_f \to \mathrm{End}_{\mathcal{F}}(V^{\otimes f})$, given by $\Omega(g_i) = R_i$ is bijective for any f, and respects the embeddings.*

Proof. Combining the map Ξ of the previous chapter with the map F of step one we know that E_f and $\mathrm{End}_{\mathcal{F}}(V^{\otimes f})$ are equal in dimension. But by compatibility of the traces and semisimplicty of both algebras, Ω is bijective. Since the braid representation π respects the embeddings

$$\mathrm{End}_{\mathcal{F}}(V^{\otimes n}) \otimes \mathrm{End}_{\mathcal{F}}(V^{\otimes m}) \to \mathrm{End}_{\mathcal{F}}(V^{\otimes (n+m)})$$

Ω must also. $\square$

Now observe that $\Omega \circ \Xi^{-1}(C_{X,X}) = t$, and since the images of $C_{X,X}$ and t under the embeddings into the larger centralizer algebras generate we have:

Theorem 5.5. *The isomorphism $\Omega \circ \Xi^{-1} : \mathrm{End}_{\mathcal{V}}(X^{\otimes (f)}) \to \mathrm{End}_{\mathcal{F}}(V^{\otimes (f)})$ induces a tensor equivalence between $\mathcal{V}$ and $\mathcal{F}$.*

Proof. We have verified that the eigenvalues of the braiding morphisms $C_{X,X}$ and t are identical. Since both Ω and Ξ respect embeddings, $\Omega \circ \Xi^{-1}$ is compatible with the tensor product rules, that is $\Omega \circ \Xi^{-1}(p_\lambda) \otimes \Omega \circ \Xi^{-1}(p_\mu)$ and $\Omega \circ \Xi^{-1}(p_\lambda \otimes p_\mu)$ are equivalent $\mathcal{F}$-morphisms for any two morphisms $p_\mu \in \mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ and $p_\lambda \in \mathrm{End}_{\mathcal{V}}(X^{\otimes m})$. $\square$

5.4 Step Three

Although the last theorem shows that the tensor product rules for $\mathcal{F}$ and $\mathcal{V}$ are the same, it does not describe the explicit correspondence between the simple objects in each category. It is given in the following theorem:

Theorem 5.6. *Let* p_λ *be a minimal idempotent in* $\mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ *with image* X_λ*. Then* $\Omega \circ \Xi^{-1}(p_\lambda)V^{\otimes n} \cong V_{\Psi(\lambda)}$.

Proof. We only need to show that this is true for the smallest n such that X_λ appears in $X^{\otimes n}$ since $\Omega \circ \Xi^{-1}$ respects equivalence of idempotents. The proof will go by induction on n. First observe that for $n = 0$ both algebras are isomorphic to $\mathbb{C}$ and so the trivial representations in each category do correspond. For $n = 1$ we again get 1-dimensional centralizer algebras so $\Omega \circ \Xi^{-1}(p_{[1]})V \cong V = V_{\Psi([1])}$. Now let $n \geq 1$ and assume that $\Omega \circ \Xi^{-1}(p_\lambda)V^{\otimes n} \cong V_{\Psi(\lambda)}$ for all $\lambda \in \Gamma(k,l)$ with $|\lambda| = n$. Let $\mu \in \Gamma(k,l)$ with $|\mu| = n+1$. By Lemma 5.1 we have that $V_{\Psi(\mu)} \subset V \otimes V_{\Psi(\lambda)}$ if and only if $X_\mu \subset X \otimes X_\lambda$, in particular if $|\lambda| = n$. Let T_n^μ be the set of $\lambda \in \Gamma(k,l)$ with $|\lambda| = n$ so that $V_{\Psi(\mu)} \subset V \otimes V_{\Psi(\lambda)}$ (recall Ψ is a bijection). Define $S_\lambda^{n+1} = \{\nu : |\nu| = n+1, X_\nu \subset X_\lambda \otimes X\}$. So by the induction hypothesis,

$$\mu \in \bigcap_{\lambda \in T_n^\mu} S_\lambda^{n+1}.$$

But any Ferrers diagram of size $n+1$ is determined by its subdiagrams of size n (see [W3] Lemma 2.11(b)), and $\Gamma(k,l)$ contains all subdiagrams of each of its members, so

$$\{\mu\} = \bigcap_{\lambda \in T_n^\mu} S_\lambda^{n+1}.$$

By the definition of T_n^μ we then have that $\Omega \circ \Xi^{-1}(p_\mu)V^{\otimes n} \cong V_{\Psi(\mu)}$ and we are done. $\square$

Chapter 6

Consequences

6.1 Failure of Positivity

In this chapter, we only need the categorical q-dimension for the object in $\mathcal{V}$ corresponding to $\lambda = [1]$:

$$Q_{[1]}(q) = \frac{[-2k]}{[1]} + 1 = \frac{q^{-2k} - q^{2k}}{q - q^{-1}} + 1$$

The crucial fact we use here is from [TuW1]: any braided category $\mathcal{O}$ with the same Grothendieck semiring as $\mathcal{V}$ has the same q-dimension as $\mathcal{V}$ up to a choice of q and the sign of $Q_{[1]}(q)$. So if we can show that $\pm Q_\lambda(q)$ is never equal to the unique positive q-character of Lemma 3.12 above for any choice of q^2 an lth root of, then we will have shown that this abstract category does not support both positivity and a braiding.

Remark 6.1. It is crucial to observe that we now allow q to be *any* power of $e^{\pi i/l}$ that is relatively prime to l. This is because we are using the fact that replacing q by $-q^{-1}$ corresponds to taking the transposed diagrams in $\Gamma(k, l)$. This does not effect the Grothendieck semiring, but does change $Q_\lambda(q)$.

By virtue of the equivalence of categories we may define the positive (normalized) q-character for $\lambda \in \Gamma(k, l)$ by $\mathrm{Dim}_q(\lambda) = \dim_q^{\Lambda_k}(\Psi(\lambda))$. The various values of $Q_\lambda(q)$ come from choosing different values of q^2 primitive lth roots of unity (which of course changes the category as well). If we set $f_\lambda(z) = Q_\lambda(e^{z\pi i/l})$ for $1 \leq z \leq l-1$ and $\gcd(l,z) = 1$, then $f_\lambda(z)$ takes on all possible values of $Q_\lambda(q)$. We may now formulate:

Theorem 6.2. *If* $2(2k+1) < l$ *then* $f_\lambda(z) \neq \mathit{Dim}_q(\lambda)$ *for any* z *with* $1 \leq z \leq l-1$ *and* $\gcd(l,z) = 1$.

Since both $Q_\lambda(q)$ and $\mathrm{Dim}_q(\lambda)$ are normalized so that their values at the trivial representation are 1, this theorem is a consequence of the following:

Lemma 6.3. *Let* $h(z) = f_{[1]}(z)$. *Then if* $2(2k+1) \leq l$ *and* $1 \leq z \leq l-1$ *with* $\gcd(l,z) = 1$ *then* $|h(z)| < \mathit{Dim}_q([1])$.

Proof. We start by showing that $h(z) < \mathrm{Dim}_q([1])$. We have that $h(z) = \frac{-\sin(2kz\pi/l)}{\sin(z\pi/l))} + 1$ and $\mathrm{Dim}_q([1]) = \frac{\sin((2k+1)\pi/l)}{\sin(\pi/l)}$. First one notes that $\mathrm{Dim}_q([1]) > 1$ and so $h(z) < \mathrm{Dim}_q([1])$ if $z \leq l/2k$. So the lemma is true for $z \in I_1 = [1, l/2k]$.

Next we make a change of variables $z \to l - z'$ in order to eliminate large z. We define

$$g(z') = h(l-z') = \frac{\sin(2kz'\pi/l)}{\sin(z'\pi/l)} + 1$$

with $1 \leq z' \leq l-1$. Using the sum expansion of $\frac{q^{2k}-q^{-2k}}{q-q^{-1}}$ we can write

$$g(z') = 1 + 2\sum_{1\leq j\leq k} \cos((2j-1)z'\pi/l).$$

By taking a derivative of $g(z')$ we find that it is a decreasing function of z' on the interval $I_2' = [2, \frac{l}{2k-1}]$, which is nonempty if $2(2k-1) \leq l$. Thus if $g(2) < \mathrm{Dim}_q([1])$ then $g(z') < \mathrm{Dim}_q([1])$ on all of I_2'. Expanding $\mathrm{Dim}_q([1])$ we compute:

$$\mathrm{Dim}_q([1]) - g(2) = 2\sum_{1\leq j\leq k} [\cos(2j\pi/l) - \cos(2(2j-1)\pi/l)].$$

Using the trigonometric formulas found in the back of any calculus book we may express each term $\cos(2j\pi/l) - \cos(2(2j-1)\pi/l)$ as $2\sin((3j-1)\pi/l)\sin((j-1)\pi/l)$.

Provided $3j-1 \leq 3k-1 \leq l$, each of these terms is positive. But we already have the stronger restriction $2(2k+1) \leq l$, thus we have $g(z') < \mathrm{Dim}_q([1])$ on I_2' that is, $h(z) < \mathrm{Dim}_q([1])$ on $I_2 = [l - \frac{l}{2k-1}, l-2]$. We check the case $z'=1$ separately:

$$\mathrm{Dim}_q([1]) - g(1) = \sum_{1 \leq j \leq k} [\cos(2j\pi/l) - \cos((2j-1)\pi/l]$$

and each term can be factored as:

$$-2\sin(\pi/2l)\sin((4j-1)\pi/2l)$$

which is always strictly negative since $4j-1 < 2l$ for all $j \leq k$.

The only remaining z to eliminate are those between I_1 and I_2. To this end we use the following estimates which come from approximating $\sin(x)$ from below by $1 - |2x/\pi - 1|$ on the interval $0 \leq x \leq \pi$:

$$h(z) < \frac{1}{\sin(z\pi/l)} + 1 < 2(2k+1)/\pi \leq \mathrm{Dim}_q([1])$$

which are valid for $z \in I_3 = [\frac{l\pi}{4(2k+1)-2\pi}, l - \frac{l\pi}{4(2k+1)-2\pi}]$ provided $2(2k+1) < l$. It is now easy to see that $[1, l-2] \cup \{l-1\} \subset I_1 \cup I_2 \cup I_3$ thus proving that $h(z) < \mathrm{Dim}_q([1])$ for any z, l, k as in the statement.

With a few modifications to this proof we can show that $-h(z) < \mathrm{Dim}_q([1])$. On I_3 our estimates are still valid. We observe that $-h(z)$ is decreasing on $[1, \frac{l}{2k-1}]$ so one need only check that $\mathrm{Dim}_q([1]) > -h(1)$, which is straightforward. By changing variables as we did above we can also eliminate $z \in [l - \frac{l}{2k}, l-2]$ using the observation that $\mathrm{Dim}_q([1]) > 1$ again. One must again check the case $z = l-1$ separately but the same basic arguement works as above except we must use the stronger condition $4k-1 \leq l$ since the factors involved are cosines. $\square$

This lemma contradicts the unitarity claim in Theorem 6.4(b) in [W1]. However, the discovery of a slight miscalculation in that paper confirms that this claim should be omitted from the statement of that theorem. Specifically, the application of the trigonometry rule

$$\sin(x) + \sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2})$$

was lacking the 2s in the denominator.

So we have shown that for no q^2 an lth root of unity does the categorical q-dimension achieve the value of the unique positive character of $Gr(\mathcal{F})$ (or $Gr(\mathcal{V})$), therefore the categorical trace-form on $\mathrm{End}_{\mathcal{F}}(W^{\otimes n})$ fails to be positive definite. We must show that no $\mathcal{B}_n$-invariant form on $\mathrm{End}_{\mathcal{F}}(W^{\otimes n})$ can be positive definite.

Lemma 6.4. *There is a simple object $V_\tau \in Ob(\mathcal{F})$ with τ an integral weight in $\mathcal{F}$ and $\dim_q(V_\tau) < 0$.*

Proof. By our Scholium 3.14 if all objects with integral weights had positive q-dimension, the simple objects with half-integral weights would have q-dimensions all positive or all negative. The first case immediately contradicts the lemma above. Observing that multiplying any character of $Gr(\mathcal{F})$ by the function $(-1)^{p(\lambda)}$ (i.e. negate all half-integral weight entries) we still have a character. But in the second case this would give us a positive character, and since we have only changed the signs, this contradicts the above lemma as well. $\square$.

Now let V_τ be a simple object with an integral weight and $\dim_q(V_\tau) < 0$ as provided by the above lemma. Then V_τ appears in some $V^{\otimes 2n}$. Let p_τ be a projection onto V_τ that is a positive self-adjoint operator. We may also choose a positive self-adjoint projection p_1 from $V^{\otimes 4n}$ onto the trivial object $\mathbb{1}$. A straightforward computation reveals that $p_1 \circ (p_\tau \otimes \mathbf{1}_{V^{\otimes 2n}}) \circ p_1 = Tr_q(p_\tau)p_1$. But $Tr_q(p_\tau) = \dim_q(V_\tau) < 0$ and the left-hand side of the above equation is a positive operator, and the right-hand side is clearly not. Since this is a purely algebraic arguement (it relies only on the braiding) it is an insurmountable obstacle to finding a positive definite $\mathcal{B}_n$-invariant form on $\mathrm{End}_{\mathcal{F}}(W^{\otimes n})$.

6.2 Modularizability

We saw in Section 3.5 that the S-matrix is not invertible for our category $\mathcal{F}$. The next best hope is that there is some quotient category of $\mathcal{F}$ that *is* modular, or more generally that there is a functor between $\mathcal{F}$ and some modular category.

This condition called *modularizability*. The scholium in Section 3.5 gives us an indication when this might be possible. A recent paper by Bruguières [Br] gives a simple criterion for determining when a premodular category is modularizable. To state his result we must give a few definitions. Let $\mathcal{O}$ be a premodular category, with simple objects $\mathcal{X}$ and S-matrix $(\Sigma_{X,Y})_{X,Y\in\mathcal{X}}$. Define $\mathcal{Z}$ to be the subset of $\mathcal{X}$ consisting of those X for which the corresponding column of the S-matrix is a multiple of the column corresponding to the identity object $\mathbb{1}\in\mathcal{X}$. That is, $X\in\mathcal{Z}$ means there is some complex number D so that $D\Sigma_{Y,X}=\Sigma_{Y,\mathbf{1}}$ for all $Y\in\mathcal{X}$. Let $\{c_{X,Y}:X,Y\in\mathcal{X}\}$ be the set of braiding operators on $\mathcal{O}$ and define

$$\mathcal{W}=\{X\in\mathcal{X}:c_{X,Y}^{-1}=c_{Y,X}\forall Y\in\mathcal{X}\}.$$

Lastly let $\dim_{\mathcal{O}}$ be the categorical dimension function (derived from the categorical trace) and θ_X the twist for the object $X\in\mathcal{X}$. Then the criterion for modularizability is:

Proposition 6.5 (Brugières). *The category $\mathcal{O}$ is modularizable if and only if for every $X\in\mathcal{Z}$ we have $X\in\mathcal{W}$, $\theta_X=\mathbf{1}_X$ and $\dim_{\mathcal{O}}(X)\in\mathbb{N}$.*

We apply this to the category $\mathcal{F}$. From scholium 3.14 we immediately see that $V_\gamma\in\mathcal{Z}$ by setting $\kappa=\mathbf{0}$. It also follows that if $k\equiv 2,3 \bmod 4$ $\mathcal{F}$ is not modularizable, since then $\dim_q(V_\gamma)=-1$. To see that in the cases where $k\equiv 1 \bmod 4$ modularizability fails we just need to compute

$$\theta_{V_\gamma}=q^{\langle\gamma+2\rho,\gamma\rangle}\mathbf{1}_{V_\gamma}.$$

One computes that $\langle\gamma+2\rho,\gamma\rangle=al/2$ where a is an odd multiple of k, so $\theta_{V_\gamma}\neq\mathbf{1}_{V_\gamma}$. When $k\equiv 0 \bmod 4$, V_γ does satisfy the conditions, so the only remaining work in order to prove that $\mathcal{Z}=\{\mathbb{1},V_\gamma\}$, that is no other simple object $V_\lambda\in\mathcal{Z}$. This will be left for later research.

6.3 Future Research

There are several questions that arose during this research that deserve attention, but were not sufficiently germane to be worked out and included. It is hoped that they will be solved in future research. We outline each one in what follows.

6.3.1 Fusion Ring Generators

The Grothendieck semiring $Gr(\mathcal{F})$ is called a fusion ring in some papers (see [Fe]). It is obviously valuable to know the explicit multiplication table for $Gr(\mathcal{F})$ since this corresponds to decomposing tensor products in the category $\mathcal{F}$. There are methods for computing parts of the multiplication tables for fusion categories associated to affine Lie algebras (see [FSS]), but these are somewhat conjectural and limited. Present research and some preliminary computations suggest that the fusion rules for tensoring with just the object V_{Λ_k} (which are given in Example 2.3) determine the rules for tensoring with any simple object. The reason is simple: for any finite-dimensional operator A whose minimal polynomial is equal to its characteristic polynomial the only operators that commute with A are the polynomials in A. Since the set of incidence matrices $\mathcal{M}$ is a commutative set of matrices, every element $M_\lambda \in \mathcal{M}$ will be a polynomial in M_{Λ_k} if we can demonstrate that M_{Λ_k} has minimal polynomial of degree $|\mathcal{M}|$. To solve for the M_λ as a polynomial in M_{Λ_k} a Gröbner basis algorithm may be employed. This has proved successful for several examples up to the rank 4, 15th root of unity case where $|\mathcal{M}| = 70$. If it can be shown that M_{Λ_k} has the property mentioned above in general, it would provide a nice theoretical tool: to check certain properties of $\mathcal{F}$ it would be enough to check them for V_{Λ_k}. It would also prove useful as a new way of computing fusion rules that could perhaps be applied to other fusion categories.

6.3.2 Classification of Fusion Categories

Lemma 3.10 indicates that the Perron-Frobenius eigenvector for the category $\mathcal{F}$ of type B is the same as for the corresponding premodular category of Lie type $\mathfrak{sp}_{2k}$ otherwise known as type C. It is easy to see that the categories have the same number of simple objects, and it should be possible to show that they are tensor equivalent using a similar analysis of the Grothendieck semiring and braiding. This would be particularly interesting in light of the fact that the categories for generic q are known to be inequivalent. There are other equivalences between fusion categories associated to affine Kac-Moody algebras and quantum groups at roots of unity (see [Fi]). It would be interesting to complete this correspondence between these three distinct derivations of fusion categories. Conjecturally, the categories derived from BMW-algebras are all equivalent to some category coming from a quantum group. But aside from the explicit equivalence described in this work and the Lie type A case, these remain conjectures. There does not appear to be any Kac-Moody algebra version of the categories of type B at odd roots of unity. Although the fixed level representations of the twisted Kac-Moody algebras of type D are labelled by the same set as the simple objects in the categories $\mathcal{F}$, no fusion product seems to exist for *any* twisted Kac-Moody algebra. It should prove to be related to the equally bad behavior of the categories described here.

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