ON A FAMILY OF NON-UNITARIZABLE RIBBON CATEGORIES

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Abstract. We consider several families of categories. The first are quotients of H. Andersen's tilting module categories for quantum groups of Lie type B at odd roots of unity. The second consists of categories of type BC constructed from idempotents in BMW-algebras. Our main result is to show that these families coincide as braided tensor categories using a recent theorem of Tuba and Wenzl. By appealing to similar results of Blanchet and Beliakova we obtain another interesting equivalence with these two families of categories and the quantum group categories of Lie type C at odd roots of unity. The morphism spaces in these categories can be equipped with a Hermitian form, and we are able to show that these categories are never unitary, and no braided tensor category sharing the Grothendieck semiring common to these families is unitarizable.

1. Introduction

The purpose of this paper is two-fold: to solve an open problem regarding the unitarity of Hermitian ribbon categories arising from quantum groups, and to make progress towards the classification of finite ribbon categories.

To any simple Lie algebra $\mathfrak{g}$ and a parameter q with q^2 a primitive ℓth root of unity one may associate a finite semisimple Hermitian ribbon category $\mathcal{F}$ derived from representations of quantum groups. A further property that $\mathcal{F}$ may have is *unitarity*, which depends on the algebra $\mathfrak{g}$ and the specific choice of q. In 1998 Wenzl [W2] showed that for $\mathfrak{g}$ of simply-laced type there is always a choice q that yields a unitary category, and for non-simply-laced types as long as ℓ is divisible by 2 (resp. 3) for types B, C and F (resp. G). It was hoped that these divisibility conditions could be removed by making a clever choice of q or changing the braiding, but whether this was possible remained a dark mystery. This was the original motivation for this paper–to explore unitarity for this family of type B, odd ℓ categories.

Among the other constructions of ribbon categories that are currently known, one of the most interesting blends ideas from operator algebras and link invariants and is essentially due to Turaev and Wenzl [TW2]. Recently Tuba and Wenzl [TuW2] studied these families of categories and were able to get a partial classification–determining the possible braiding and monoidal structures from the Grothendieck semiring. We use their result to identify the aforementioned family of Lie type B, odd ℓ quantum group categories with certain Turaev-Wenzl categories of ortho-symplectic BC type at the level of braided tensor categories. Similar results were obtained by A. Beliakova and C. Blanchet in [BB]. The main equivalence we establish is just an extension of an equivalence Beliakova and Blanchet observed to spin modules. Combining their results with ours we get as a corollary a rank-level type duality between the Lie type B and C quantum group categories at odd roots of unity (see Corollary 6.6).

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By a thorough (but elementary) analysis of characters of the Grothendieck semirings of these categories, we are able to show that no Hermitian ribbon category with the same tensor product rules as these categories can be unitary. Thus we answer the original question of unitarity for both quantum groups of Lie types B and C at odd roots of unity.

The organization of this paper is as follows. In Section 2 we define the categorical terms of the subject and mention a few results germane to the discussion. In Sections 3 and 4 we describe the structure of the family of quantum group categories we are concerned with and analyze the Grothendieck semiring and characters. This sets the stage Section 5 in which we consider the representations of the braid group on morphism spaces and the second family of categories we consider. In Section 6 we establish the equivalence between these two families of categories. In Section 7 we apply this equivalence to prove the failure of unitarity.

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2. Ribbon Categories

2.1. **Axioms.** In this subsection we outline the relevant categorical axioms. We follow the paper [TW2], and refer to that paper or the books by Turaev [Tur] or Kassel [K] for a complete treatment.

Let $\mathcal{O}$ be a category defined over a subfield $k \subset \mathbb{C}$. The following axioms are satisfied by a semisimple Hermitian ribbon category.

1. A **monoidal category** has a tensor product $\otimes$ and an identity object $\mathbb{1}$ satisfying the triangle and pentagon axioms. These guarantee that the tensor product is associative (at least up to isomorphism) and that

$$\mathbb{1} \otimes X \cong X \otimes \mathbb{1} \cong X$$

for any object X. We usually assume our categories are *strict*, that is, that the associativity isomorphisms and the isomorphisms above are the identity.

2. A category is **rigid** if there is a dual module X^* for each object X and morphisms

$$b_X : \mathbb{1} \to X \otimes X^*, d_X : X^* \otimes X \to \mathbb{1}$$

satisfying

$$(\mathrm{Id}_X \otimes d_X)(b_X \otimes \mathrm{Id}_X) = \mathrm{Id}_X \tag{1}$$

$$(d_X \otimes \mathrm{Id}_{X^*})(\mathrm{Id}_{X*} \otimes b_X) = \mathrm{Id}_{X^*}. \tag{2}$$

3. An **Ab-category** is one in which all morphism spaces are $\mathbb{C}$-vector spaces and the composition and tensor product of morphisms are bilinear.
4. A **semisimple** category has the property that every object X is isomorphic to a finite direct sum of *simple* objects–that is, objects X_i with $\mathrm{End}(X_i) \cong \mathbb{C}$–and that the simple

objects satisfy Schur's Lemma: $\dim \mathrm{Hom}(X_i, X_j) \in \{0,1\}$. $\mathcal{O}$ is called **finite** if there are finitely many isomorphism classes of simple objects.

5. A **braiding** is a family of isomorphisms

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

satisfying

$$c_{X,Y\otimes Z} = (\mathrm{Id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{Id}_Z) \tag{3}$$

$$c_{X\otimes Y,Z} = (c_{X,Z} \otimes \mathrm{Id}_Y)(\mathrm{Id}_X \otimes c_{Y,Z}) \tag{4}$$

6. A **twist** consists of isomorphisms

$$\theta_X : X \to X$$

To be compatible with the braiding and duality we must have:

$$\theta_{X\otimes Y} = c_{Y,X}c_{X,Y}(\theta_X \otimes \theta_Y) \tag{5}$$

$$\theta_{X^*} = (\theta_X)^* \tag{6}$$

A rigid category is called *balanced* if it has a twist.

7. A **Hermitian** category has a conjugation:

$$\dagger : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X)$$

such that $(f^\dagger)^\dagger = f$, $(f\otimes g)^\dagger = f^\dagger \otimes g^\dagger$ and $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$. On $\mathbb{C}$, $\dagger$ must also act as the usual conjugation. Furthermore, $\dagger$ must also be compatible with the other structures present i.e.

$$(c_{X,Y})^\dagger = (c_{X,Y})^{-1} \tag{7}$$

$$(\theta_X)^\dagger = (\theta_X)^{-1} \tag{8}$$

$$(b_X)^\dagger = d_X c_{X,X^*}(\theta_X \otimes \mathrm{Id}_{X^*}) \tag{9}$$

$$(d_X)^\dagger = (\mathrm{Id}_{X^*} \otimes \theta_X^{-1})(c_{X^*,X})^{-1} b_X \tag{10}$$

Remark 2.1. For any $f \in \mathrm{Hom}(X,Y)$ we define $f^* \in \mathrm{Hom}(Y^*, X^*)$ by:

$$f^* = (d_Y \otimes \mathrm{Id}_{X^*})(\mathrm{Id}_{Y^*} \otimes f \otimes \mathrm{Id}_{X^*})(\mathrm{Id}_{Y^*} \otimes b_X).$$

Remark 2.2. We will often consider categories satisfying some subset of these axioms; for example a *braided tensor category* satisfies axioms 1-5.

2.2. **General Consequences.** The categorical axioms above supply us with several useful tools for studying these categories. The following results are found in the references mentioned above or in [OW] and [TuW2].

2.2.1. *Categorical Trace.* In any semisimple ribbon category one defines a categorical trace for any morphism $f \in \mathrm{End}_{\mathcal{O}}(X)$:

$$Tr_{\mathcal{O}}(f) = d_X c_{X,X^*}((\theta_X f) \otimes \mathrm{Id}_{X^*}) b_X : \mathbb{1} \to \mathbb{1}. \tag{11}$$

One defines the categorical dimension of an object X by:

$$\dim_{\mathcal{O}}(X) := Tr_{\mathcal{O}}(\mathrm{Id}_X).$$

It is often useful to normalize the trace so that the trace of the identity morphism Id_X has trace 1 where X is any object. This is achieved by setting $tr_{\mathcal{O}}(f) = Tr_{\mathcal{O}}(f)/\dim_{\mathcal{O}}(X)$ for any $f \in \mathrm{End}(X)$.

The expected properties of the trace go through and are by now well-known.

Lemma 2.3. *(a)* $Tr_{\mathcal{O}}(f \circ g) = Tr_{\mathcal{O}}(g \circ f)$ *when the composition and trace are defined.*
(b) $Tr_{\mathcal{O}}(f \otimes g) = Tr_{\mathcal{O}}(f)Tr_{\mathcal{O}}(g)$.
(c) $\dim_{\mathcal{O}}(X) \neq 0$ *if* X *is simple.*

A proof of the following important result can be found in [OW].

Lemma 2.4. *Let* $\mathcal{O}$ *be a semisimple ribbon category, and* X *and* Y *be simple objects in* $\mathcal{O}$, *with* $p \in \mathrm{End}(X \otimes X^*)$ *the projection onto the subobject of* $X \otimes X^*$ *isomorphic to* $\mathbb{1}$, *and* $a \in \mathrm{End}(Y \otimes X)$. *Then*

$$(\mathrm{Id}_Y \otimes p)(a \otimes \mathrm{Id}_{X^*})(\mathrm{Id}_Y \otimes p) = \frac{Tr_{\mathcal{O}}(a)}{\dim_{\mathcal{O}}(Y)\dim_{\mathcal{O}}(X)}(\mathrm{Id}_Y \otimes p)$$

The proof is an exercise in the so-called graphical calculus of ribbon categories. For an explicit formula for p one may take $1/\dim_{\mathcal{O}}(X) b_X b_X^{\dagger}$ (which is defined regardless of the existence of a conjugation in the category).

Lemma 2.4 has the following specialization known as the *Markov property* (see [TuW2]):

Lemma 2.5. *If* $a \in \mathrm{End}(X^{\otimes n})$ *and* $m \in \mathrm{End}(X^{\otimes 2})$, *then*

$$tr((a \otimes \mathrm{Id}_X) \circ (\mathrm{Id}_X^{\otimes(n-1)} \otimes m)) = tr(a)tr(m).$$

2.2.2. *Representations of* $\mathbb{C}\mathcal{B}_n$. The braiding axiom implies that the operators $c_1 := c_{X,X} \otimes \mathrm{Id}_X$ and $c_2 := \mathrm{Id}_X \otimes c_{X,X}$ in $\mathrm{End}_{\mathcal{O}}(X^{\otimes 3})$ satisfy the braid relation $c_1c_2c_1 = c_2c_1c_2$, and hence we obtain representations of the group algebra of the braid group $\mathbb{C}\mathcal{B}_n \to \mathrm{End}_{\mathcal{O}}(X^{\otimes n})$ by sending

$$\sigma_i \to c_i := \mathrm{Id}_X^{\otimes(i-1)} \otimes c_{X,X} \otimes \mathrm{Id}_X^{\otimes(n-i-1)}$$

One may also define a representation of $\mathbb{C}\mathcal{B}_n$ on the vector space $\mathrm{End}_{\mathcal{O}}(X^{\otimes n})$ by composing with c_i. Here σ_i is the standard generator of $\mathcal{B}_n$ as shown in Figure 1.

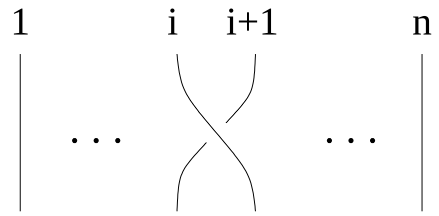

FIGURE 1. The generator σ_i

Tuba and Wenzl in [TuW1] succeeded in classifying all representations of B_3 of dimension ≤ 5 by the eigenvalues of the image of σ_1 and the scalar by which the center of B_3 acts. This becomes quite useful as the structure of the representation of $\mathbb{C}\mathcal{B}_n$ on $\mathrm{End}(X^{\otimes n})$ is already essentially determined by considering $n = 3$.

2.2.3. *Grothendieck Semiring.* We also have the Grothendieck semiring $Gr(\mathcal{O})$ of a finite semisimple monoidal category $\mathcal{O}$. Let X_i $1 \leq i \leq m$ be a complete set of representatives of distinct isomorphism classes of simple objects in $\mathcal{O}$. Define $N_{ij}^k := \dim \mathrm{Hom}(X_k, X_i \otimes X_j)$ so that $X_i \otimes X_j \cong \bigoplus N_{ij}^k X_k$. The braiding ensures that $N_{ij}^k = N_{ji}^k$, and the Grothendieck semiring is the commutative quotient ring:

$$Gr(\mathcal{O}) := \mathbb{Z}[X_i]/\langle X_iX_j - \sum_k N_{ij}^k X_k : 1 \leq i,j,k \leq m\rangle \tag{12}$$

Remark 2.6. The relations in $Gr(\mathcal{O})$ are often called *fusion rules*, particularly in the physics literature.

The left action of $Gr(\mathcal{O})$ on itself is given in the basis $\{X_i\}$ is $X_i \to N_i$ where $(N_i)_{kj} = N_{ij}^k$. By commutativity the simple representations of $Gr(\mathcal{O})$ are 1-dimensional, and we can study the character theory.

Definition 2.7. A *character* for $Gr(\mathcal{O})$ is any representation $f : Gr(\mathcal{O}) \to \mathbb{C}$ that satisfies

$$f(X_i)f(X_j) = \sum_k N_{ij}^k f(X_k) \tag{13}$$

We have already seen one character of $Gr(\mathcal{O})$, namely the function $\dim_{\mathcal{O}}$. Observe that for any character f of $Gr(\mathcal{O})$ the vector $\mathbf{f} := \sum_i f(X_i)X_i$ is a simultaneous eigenvector of the set of matrices $\mathcal{M} := \{N_i\}$. In particular there can be at most $|\{X_i\}|$ inequivalent characters.

2.3. **Self-Dual Categories.** For convenience of notation, we make the following definition:

Definition 2.8. A *self-dual* category is one in which every object is isomorphic to its dual object.

All of the categories that we will consider in detail will be self-dual. This gives $Gr(\mathcal{O})$ a much simpler structure: the N_{ij}^k are totally symmetric in the i, j and k. Lemma 2.4 has a stronger consequence in the self-dual case (see [TuW2]):

Corollary 2.9. *Suppose $X^{\otimes 2} \cong \bigoplus_i X_i$ in a self-dual semisimple ribbon category $\mathcal{O}$, and we have a basis of mutually annihilating idempotents $p_j \in \mathrm{End}(X^{\otimes 2})$ so that $p_j X^{\otimes 2} \cong X_j$ and $X_1 \cong \mathbb{1}$. Then*

$$(\mathrm{Id}_X \otimes p_1)(p_j \otimes \mathrm{Id}_X)(\mathrm{Id}_X \otimes p_1) = \frac{\dim_{\mathcal{O}}(X_j)}{(\dim_{\mathcal{O}}(X))^2}(\mathrm{Id}_X \otimes p_1)$$

2.3.1. *Unitary Categories.* In a semisimple Hermitian ribbon category, the form $\langle f, g\rangle := Tr_{\mathcal{O}}(f^\dagger g)$ on $\mathrm{Hom}(X, Y)$ is Hermitian. Since $(c_{X,Y})^\dagger = (c_{X,Y})^{-1}$ the form $\langle\ ,\ \rangle$ on $\mathrm{End}(X^{\otimes n})$ is preserved by the action of $\mathbb{C}\mathcal{B}_n$:

$$\langle c_i f, c_i g\rangle = Tr_{\mathcal{O}}(f^\dagger (c_i)^{-1} c_i g) = \langle f, g\rangle.$$

So if $\langle\ ,\ \rangle$ is positive definite on $\mathrm{End}(X^{\otimes n})$ for all n then the representations of $\mathbb{C}\mathcal{B}_n$ is on the Hilbert spaces $\mathrm{End}(X^{\otimes n})$ $n \geq 1$ are unitary. For this reason such categories are called *unitary.*

Definition 2.10. A braided tensor category $\mathcal{O}$ is called *unitarizable* if there is a Hermitian form on the morphism spaces with respect to which the representations of $\mathbb{C}\mathcal{B}_n$ are unitary.

If $\mathcal{O}$ is a finite semisimple Hermitian ribbon Ab-category then the positivity of the form $\langle\ ,\ \rangle$ is determined by positivity on the idempotents $p_i \in \mathrm{End}(X^{\otimes n})$ where $p_i X^{\otimes n} \cong X_i$, since $\mathrm{End}(X^{\otimes n})$ is a direct sum of full matrix algebras by semisimplicity. Assume that $\mathcal{O}$ is self-dual. Then we can choose the idempotents so that $(p_i)^\dagger = p_i$ (self-adjoint) and then we have that

$$\langle p_i, p_i\rangle = Tr_{\mathcal{O}}(p_i) = \dim_{\mathcal{O}}(X_i)/(\dim_{\mathcal{O}}(X))^n$$

by the lemmas above. In particular, if $\dim_{\mathcal{O}}(X_i) > 0$ for all simple objects X_i, then $\mathcal{O}$ is unitary.

Theorem 2.11. *Let $\mathcal{O}$ be a semisimple self-dual Hermitian ribbon category. Suppose that some simple object X_i appears in $Y^{\otimes 2n}$ with $\dim_{\mathcal{O}}(X_i) < 0$. Then the category $\mathcal{O}$* cannot *be unitary for* any *$\mathbb{C}\mathcal{B}_n$ invariant Hermitian form.*

Proof. Assume to the contrary that there is some $\mathbb{C}B_k$-invariant positive Hermitian form $(\ ,\ )$ on each $\mathrm{End}(Y^{\otimes k})$. Then choose positive projections $p, p_i \in \mathrm{End}(Y^{\otimes 2n})$ with images isomorphic to $\mathbb{1}$ and X_i respectively. By applying the corollary to Lemma 2.4 in the self-dual case for $X = Y^{\otimes 2n}$ we compute:

$$(\mathrm{Id}_Y^{\otimes 2n} \otimes p)(p_i \otimes \mathrm{Id}_Y^{\otimes 2n})(\mathrm{Id}_Y^{\otimes 2n} \otimes p) = \frac{\dim_{\mathcal{O}}(X_i)}{(\dim_{\mathcal{O}}(Y))^{2n}}(\mathrm{Id}_Y^{\otimes 2n} \otimes p).$$

But observe that the left-hand side of this equality is a positive operator, whereas the right-hand side is a negative operator as $\dim_{\mathcal{O}}(X_i) < 0$. □

3. Ribbon Categories from Quantum Groups

In this section we discuss the ribbon categories derived from the quantum groups at roots of unity. The construction of the category is by now quite well-known and we will omit the details. We content ourselves to refer the interested reader to: Jantzen's book [Ja] for an introduction to quantum groups and to illuminate the way through Lusztig's book [Lu] on the same, the papers of Andersen and Paradowski [A], [AP] for the categories of tilting modules and their semisimple quotient, and chapters 9-11 of the book by Chari and Pressley [ChPr] for certain cases of the whole construction.

3.1. **Notation and Outline.** We will need some notation in order to proceed. Let $\mathfrak{g}$ be a simple Lie algebra of rank k. We have:

- the root system Φ embedded in $\mathbb{R}^k$
- the Cartan matrix $A = (a_{ij})$
- a root basis $\Pi = \{\alpha_i\}_{i=1}^k$
- positive roots $\Phi_+ = \mathbb{N}\Pi \cap \Phi$
- root lattice $Q = \mathbb{Z}\Pi$
- a normalized bilinear form $\langle\ ,\ \rangle$ so that $2\langle\alpha_i, \alpha_j\rangle/\langle\alpha_j, \alpha_j\rangle = a_{ij}$ and $\langle\alpha, \alpha\rangle = 2$ for short roots.
- coroot basis $\check{\Pi} = \{\check{\alpha}_i\}_{i=1}^k$, $\check{\alpha}_i := 2\alpha_i/\langle\alpha_i, \alpha_i\rangle$
- coroot system $\check{\Phi}$
- positive coroots $\check{\Phi}_+$
- Weyl group W generated by: $s_i(v) := v - \langle v, \check{\alpha}_i\rangle\alpha_i$
- fundamental weights Λ_i dual to the coroot basis via $\langle\ ,\ \rangle$
- weight lattice $P = \mathbb{Z}\{\Lambda_i\}$
- dominant Weyl chamber $C := \mathbb{R}^+\{\Lambda_i\}$ containing the dominant weights $P_+ := \mathbb{N}\{\Lambda_i\}$.

Drinfeld [D] and Jimbo defined a quantum group as a q-deformation $\mathcal{A}_q(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$ where the base field is $Q(q)$ with q an indeterminate. The finite dimensional representations of $\mathcal{A}_q(\mathfrak{g})$ are integral and highest weight and the Grothendieck semiring of this representation category is isomorphic to that of $\mathfrak{g}$ itself. However, if we try to specialize q to a root of unity $\mathcal{A}_q(\mathfrak{g})$ is no longer well-defined. Now let q^2 be a primitive ℓth root of unity, that is, $q = e^{z\pi i/\ell}$ with $\gcd(z, \ell) = 1$. Lusztig's "modified form" of $\mathcal{A}_q(\mathfrak{g})$ denoted $U_q\mathfrak{g}$ is well-defined for any complex $q \notin \{-1, 0, 1\}$. That $U_q\mathfrak{g}$ is a ribbon Hopf algebra follows from the work of Drinfeld, Lusztig and others, see [BK] for details. For each $\lambda \in P_+$ one constructs a *Weyl module* V_λ of $U_q(\mathfrak{g})$ by restricting the corresponding highest weight $\mathcal{A}_q(\mathfrak{g})$-module to $U_q(\mathfrak{g})$ and specializing the parameter q to the chosen root of unity. The generators of $U_q\mathfrak{g}$ act on Weyl modules by matrices with entries in $\mathbb{Z}[q^{\pm 1}]$. The Weyl modules are not all irreducible or even indecomposable. To remedy this, H. Andersen [A] defined a category $\mathcal{T}$ of *tilting modules* that have the following key properties:

1. For each $\lambda \in P_+$ there is a unique indecomposable tilting module T_λ.
2. The set $\mathcal{I} = \{T \in \mathcal{T} : \dim_{\mathcal{T}}(T) = 0\}$ is a tensor ideal.
3. There are finitely many indecomposable $T_\lambda \notin \mathcal{I}$. These are irreducible and isomorphic to the corresponding Weyl module.
4. The category $\mathcal{F} = \mathcal{T}/\mathcal{I}$ defined by taking the quotient of the morphisms in $\mathcal{T}$ by the radical of $Tr_{\mathcal{T}}$ is a semisimple ribbon category.

Tilting modules can be realized as direct sums of submodules of tensor powers of the *fundamental module(s)*. A fundamental module is one that generates the category $Rep(U_q\mathfrak{g})$ generically, that is every irreducible module appears in some tensor power.

3.2. **The Category $\mathcal{F}$.** We can describe the category $\mathcal{F}$ as follows. Let d be the ratio of the squared length of a long root in $\mathfrak{g}$ to the squared length of a short root. If q^2 is a primitive ℓth root of unity with ℓ divisible by d then let θ be the highest root of Φ, if ℓ is coprime to d then let θ be the highest short root. Then the simple objects of $\mathcal{F}$ are isomorphic to Weyl modules V_λ with $\lambda \in C_\ell := \{\mu \in P_+ : \langle \mu + \rho, \check{\theta}\rangle < \ell\}$, here ρ is half the sum of the positive roots $\alpha \in \Phi_+$. In fact the indecomposable tilting modules T_μ that are isomorphic to irreducible Weyl modules are labeled by

$$\mu \in \overline{C_\ell} := \{\mu \in P_+ : \langle \mu + \rho, \check{\theta}\rangle \leq \ell\}.$$

To avoid degeneracies we always assume that the rank k and ℓ are such that $\rho + \Lambda_1 \in C_\ell$, where Λ_1 is the dominant weight of the defining representation of $\mathfrak{g}$. By taking the convex hull of the set C_ℓ we obtain the *fundamental Weyl alcove* denoted by D.

3.2.1. *Affine Weyl Group.* The dominant Weyl chamber C is described as the fundamental domain of the Weyl group W containing ρ, the fundamental Weyl alcove D can be similarly described:

Definition 3.1. Denote the affine reflection in $\mathbb{R}^k$ through the hyperplane $\{x \in \mathbb{R}^k : \langle x, \check{\theta}\rangle = \ell\}$ by t_ℓ. If we adjoin t_ℓ to the Weyl group W we get the *affine* Weyl group W_ℓ. Explicitly $t_\ell(\lambda) = \lambda + (\ell - \langle \lambda, \check{\theta}\rangle)\theta$.

We must define a slightly different action of W_ℓ on P than the usual one inherited from Euclidean $\mathbb{R}^k$. For $w \in W_\ell$ and $s \in \mathbb{R}^k$ define the "dot action" $w \cdot x := w(x+\rho) - \rho$. Then D is the fundamental domain of the dot action of W_ℓ on $\mathbb{R}^k$ and of course $C_\ell = D \cap P_+$. The elements of W have a natural signature ε depending on the number of simple reflections s_i in any decomposition. If we assign $\varepsilon(t_\ell) = -1$ then this extends the signature function to W_ℓ.

We now proceed to describe the categorical structure on $\mathcal{F}$.

3.2.2. *Monoidal Structure.* $\mathcal{F}$ inherits a monoidal structure from the comultiplication and counit in the Hopf algebra $U_q\mathfrak{g}$.

3.2.3. *Duality.* The dual module of a simple Weyl module V_λ is the ordinary vector space dual with the action of $U_q\mathfrak{g}$ defined via the antipode. V_λ^* is also a Weyl module with highest weight equal to $-w_0(\lambda)$ where w_0 is the longest element in the Weyl group with respect to Bruhat order. One checks that $-w_0(\lambda) \in C_\ell$. The rigidity morphisms are defined

$$b_V : 1 \to \sum_i v_i \otimes v^i$$

and

$$d_V : f \otimes v \to f(v)$$

where v_i is a basis of V and v^i is the dual basis (of V^*).

3.2.4. *Braiding.* Lusztig [Lu] showed that the universal R-matrix in $U_q\mathfrak{g}$ specializes to the root of unity case. Composing with the flip operator σ we get well-defined operators $\check{R}_{V,W}$ for any objects $V, W \in \mathcal{F}$. These do satisfy the braiding identities. We have the very useful (see [D]):

Proposition 3.2 (Drinfeld)**.** *If V_λ and V_μ are simple Weyl modules such that V_ν appears in $V_\lambda \otimes V_\mu$ then one has:*

$$\check{R}_{\mu,\lambda}\check{R}_{\lambda,\mu}\mid_{V_\nu} = q^{c_\nu - c_\lambda - c_\mu}\mathrm{Id}_{V_\nu}$$

where $c_\gamma := \langle \gamma + 2\rho, \gamma\rangle$.

To conform with our original notation we will denote the morphisms $\check{R}_{V,W}$ by $c_{V,W}$.

3.2.5. *Twist.* It also follows from the work of Drinfeld that there is a *universal Casimir* operator in $U_q\mathfrak{g}$ that provides $\mathcal{F}$ with a twist. For a simple object V_λ the twist θ_λ acts by the constant q^{c_λ} where c_λ is as above.

3.2.6. *Ab-structure.* The spaces $\mathrm{Hom}(V, W)$ are quotients of the vector spaces of intertwining operators in the category $\mathcal{T}$, so they are themselves $\mathbb{C}$-vector spaces.

3.2.7. *Finite Semisimplicity.* Andersen's ([A]) main result shows that $\mathcal{F}$ is semisimple, as we have taken the quotient by the radical part of the category, and all other necessary properties are inherited from the category $\mathcal{T}$. Only finitely many isomorphism classes of simple objects V_λ survive in the quotient.

3.2.8. *Hermitian Form.* Kirillov Jr. [Ki] succeeded in defining a conjugation on the category $\mathcal{F}$. In this paper we are only concerned with the existence of one, so we will not go into details.

3.2.9. *Categorical Trace.* With all of the above structure, $\mathcal{F}$ is a ribbon category and hence has a trace. We can compute the value of $\dim_{\mathcal{T}}$ explicitly on the objects $T_\mu \cong V_\mu$, $\mu \in \overline{C_\ell}$:

$$\dim_{\mathcal{T}}(V_\mu) = \prod_{\alpha\in\Phi_+} \frac{[\langle \mu + \rho, \alpha\rangle]}{[\langle \rho, \alpha\rangle]}$$

where $[n] := \frac{q^n - q^{-n}}{q - q^{-1}}$. This follows from the proof of the Weyl dimension formula in the classical theory. Since $\theta \in \Phi_+$ one sees that $\dim_{\mathcal{T}}(V_\mu) = 0$ for $\mu \in (\overline{C_\ell} \setminus C_\ell)$. By construction $\dim_{\mathcal{T}}$ vanishes on the ideal $\mathcal{I}$ so the categorical dimension $\dim_{\mathcal{F}}$ coincides with $\dim_{\mathcal{T}}$ on the quotient.

3.2.10. *Grothendieck Semiring.* The Grothendieck semiring $Gr(\mathcal{F})$ is a quotient of $Gr(Rep(\mathfrak{g}))$. The structure constants of $Gr(\mathcal{F})$ are W_ℓ-antisymmetrizations of those of $Gr(Rep(U_q\mathfrak{g}))$ for q generic (which are the same as those of $Gr(Rep(\mathfrak{g}))$).

Remark 3.3. The proposition that follows was proved for weights in the *root* lattice by Andersen and Paradowski ([AP], Prop. 3.20), as the quantum group studied there is constructed from the *adjoint* root datum whereas we want to use the *simply connected* root datum (see [Lu], Chapter 2). So in particular one must justify the extension of this result to those weights not in the root lattice, that is, the half-integer weights. However, the argument presented in [AP] relies only upon results in [A] (which are valid for ℓ coprime to the nonzero entries of the Cartan matrix, in particular for

Lie type B quantum groups with ℓ odd: see Section 1 of [A]) and therefore carries over word-for-word to the case at hand. In fact, the only results cited by Andersen and Paradowski for which they do not give an explicit reference in [A] are the linkage principle and their "quantum version of Proposition 2.5". These are found in [A] statement (1.2) and Theorem 2.5 respectively. With this justification we attribute the proposition below to Andersen and Paradowski.

If $m_{\lambda\mu}^{\nu} = \dim \mathrm{Hom}_{U_q\mathfrak{g}}(V_\nu, V_\lambda \otimes V_\mu)$ for $\lambda, \mu, \nu \in P_+$ (that is, $m_{\lambda\mu}^{\nu}$ are the classical weight multiplicities), then we have (see [AP], Prop. 3.20):

Proposition 3.4 (Andersen-Paradowski). *For simple objects V_λ, V_μ in the category $\mathcal{F}$,*

$$N_{\lambda\mu}^{\nu} = \sum_{w \in W_\ell: \quad w\cdot\nu \in P_+} \varepsilon(w) m_{\lambda\mu}^{w\cdot\nu}$$

where $N_{\lambda\mu}^{\nu} := \dim \mathrm{Hom}_{\mathcal{F}}(V_\nu, V_\lambda \otimes V_\mu)$.

Observe that if $\langle \nu + \rho, \check{\theta} \rangle = \ell$ (i.e. $\nu \in \overline{C_\ell} \setminus C_\ell$ and $\dim_{\mathcal{T}}(V_\nu) = 0$) then

$$t_\ell \cdot \nu = t_\ell(\nu + \rho) - \rho = (\nu + \rho) + (\ell - \langle (\nu + \rho), \check{\theta} \rangle)\theta - \rho = \nu$$

so the antisymmetrization above gives $N_{\lambda\mu}^{\nu} = 0$ as expected.

4. Type B at Odd ℓ Categories

Observe that the construction of the categories above depend on two choices: a Lie algebra $\mathfrak{g}$ and a root of unity q^2. We now specialize to the categories we will study in detail: that is, the Lie algebra $\mathfrak{g} \cong \mathfrak{so}_{2k+1}$, and q^2 a primitive ℓth root of unity, ℓ odd. For a fixed ℓ and k, we denote by $\mathfrak{F}$ the family of ribbon categories constructed as above from $\mathfrak{so}_{2k+1}$ with q^2 any primitive ℓth root of unity. A fixed member of this family will be denoted by $\mathcal{F}$.

4.1. **Type B Data.** Let $\{\varepsilon_i\}$ be the standard basis for $\mathbb{R}^k$. We fix a root basis

$$\Pi = \{\alpha_i\}_1^k = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{k-1} - \varepsilon_k, \varepsilon_k\}$$

so the root lattice $Q = \mathrm{span}_{\mathbb{Z}}\{\alpha_\mathrm{i}\}_1^\mathrm{k}$ is just $\mathbb{Z}^k$. We also record that the set of positive roots is

$$\Phi_+ = \{\varepsilon_s \pm \varepsilon_t, \varepsilon_u : s < t\}.$$

The form $\langle \ , \ \rangle$ is twice the usual dot product on $\mathbb{R}^k$ so that the square length of long roots is 4, and 2 for short roots. Thus the coroot basis $\check{\Pi} = \{\check{\alpha}\}$ has

$$\check{\alpha}_i = \begin{cases} \frac{1}{2}(\varepsilon_i - \varepsilon_{i+1}) & i = 1, \ldots, k-1 \\ \varepsilon_k & i = k \end{cases}$$

Note that classically, the coroots for type B are the roots of type C, but here we must take care as the normalization of the form is not the classical one. We will see where this leads to subtleties later. The Weyl group W is the semi-direct product of S_k and $(\mathbb{Z}_2)^k$ and acts on $\mathbb{R}^k$ via permutations and sign changes.

For our choice of a root basis we have the following *fundamental weights*:

$$\Lambda_i = \begin{cases} \sum_{1 \leq j \leq i-1} \varepsilon_j & i \leq k-1 \\ \frac{1}{2} \sum_{1 \leq j \leq k} \varepsilon_i & i = k \end{cases}$$

and the dominant weights $P_+ = \mathrm{span}_{\mathbb{N}}\{\Lambda_j\}_1^k$. The weight lattice $P = \mathrm{span}_{\mathbb{Z}}\{\Lambda_j\}_1^k$ is then seen to be $\mathbb{Z}^k \bigcup (\Lambda_k + \mathbb{Z}^k)$. For convenience of notation we introduce the function on P:

$$p(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \mathbb{Z}^k \\ -1 & \text{if } \lambda \in (\Lambda_k + \mathbb{Z}^k) \end{cases}$$

We refer to a weight λ as integral, resp. half-integral, if $p(\lambda) = 1$, resp. $p(\lambda) = -1$. The weights are usually represented as k-tuples, e.g. $\Lambda_k = (1/2, \ldots, 1/2)$.

For type B we have that $w_0 = -1$, that is, the element of the Weyl group that changes the sign of each coordinate. Since the weight of V_λ^* is $-w_0(\lambda) = \lambda$, all modules are self-dual in the present case.

4.2. **Classical Representation Theory, Abridged.** As we noted above, for generic parameters q, we have $Gr(Rep(U_q\mathfrak{g})) \cong Gr(Rep(\mathfrak{g}))$, and $Gr(\mathcal{F})$ is a quotient of these rings, so in this subsection we will summarize the necessary facts from the representation theory of the algebra $U\mathfrak{so}_{2k+1}$. This material can be found in any introductory text on Lie groups, such as [GWa] or [Hu], and goes back at least to Weyl [Wy].

The irreducible finite-dimensional integral highest weight modules of $\mathfrak{so}_{2k+1}$ are in one-to-one correspondence with the elements of P_+. Each irreducible integral highest weight module V_λ has a multiset of weights $P(\lambda)$ which correspond to the weight-space decomposition of V_λ with respect to the action of the Cartan subalgebra. The multiset $P(\lambda)$ lies in the ball of radius $|\lambda|$ (ordinary Euclidean distance) centered at the origin, and the weights in the W-orbit of λ appear with multiplicity one. The other weights are of the form $\lambda - \alpha$ for some $\alpha \in Q$. To decompose the tensor product of two irreducible modules V_λ and V_μ one looks at the intersection $\{\nu = \mu + \kappa : \kappa \in P(\lambda)\} \bigcap P_+$ which contains the dominant weights of the irreducible submodules

$$P_+(V_\lambda \otimes V_\mu) = \{\nu \in P_+ : V_\nu \subset V_\lambda \otimes V_\mu\}.$$

We do not formulate the precise algorithm to determine which V_ν do occur nor the multiplicities, but we can say that the irreducible module $V_{\mu+w(\lambda)}$ appears with multiplicity one, where w is any element in the Weyl group such that $w(\lambda) + \mu \in P_+$. (This follows from the outer multiplicity formula, see e.g. [GWa] Corollary 7.1.6). Moreover, $P_+(V_\lambda \otimes V_\mu)$ is contained in the ball of radius $|\lambda|$ centered at μ, and $p(\nu) = p(\lambda)p(\mu)$ for any $\nu \in P_+(V_\lambda \otimes V_\mu)$. In other words, all weights of simple submodules of $V_\lambda \otimes V_\mu$ are integral if λ and μ are both integral or half-integral, and half-integral otherwise.

4.3. **Structure Constants of $Gr(\mathcal{F})$.** Recall the left regular representation of $Gr(\mathcal{F})$ from 2.2.3 and denote the images of the generators by N_λ, $\lambda \in C_\ell$. In general it is not easy to compute the entries $N_{\lambda\mu}^\nu$ of the matrices N_λ as it is already difficult to compute the classical multiplicities $m_{\lambda\mu}^\nu$; however, for our analysis we only require two explicit decomposition rules–both of which were already known to Brauer in the 1940s. We begin with the decomposition rules for tensoring with the generating module V_{Λ_k}.

Example 4.1. We have that V_{Λ_k} is a minuscule representation (all weights are conjugate under the Weyl group) the simple decomposition as a $\mathfrak{so}_{2k+1}$-module is:

$$V_{\Lambda_k} \otimes V_\lambda \cong \bigoplus_{W_k^\lambda} V_{\lambda + w(\Lambda_k)}$$

where $W_k^\lambda = \{w \in W : \lambda + w(\Lambda_k) \in P_+\}$ Note that $W(\Lambda_k) = \{\frac{1}{2}(\pm 1, \ldots, \pm 1)\}$, so all $\lambda + w(\Lambda_k)$ are in $\overline{C_\ell}$, so the W_ℓ-antisymmetrization has the effect of discarding the $V_{\lambda+w(\Lambda_k)} \in \overline{C_\ell} \setminus C_\ell$ and leaving all other objects alone. That is, for $\lambda, \nu \in C_\ell$

$$N_{\Lambda_k \lambda}^{\nu} = \begin{cases} 1 & \text{if } \nu = \lambda + w(\Lambda_k) \text{ some } w \in W \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

Lemma 4.2. V_{Λ_k} *generates* $\mathcal{F}$.

Proof. We will show that there exists an odd integer s such that *every* simple object in $\mathcal{F}$ appears in $V_{\Lambda_k}^{\otimes s}$ or $V_{\Lambda_k}^{\otimes s+1}$. Every weight $\lambda \in C_\ell$ can be expressed as a sum of weights in $W(\Lambda_k)$, so every V_λ appears in some tensor power of V_{Λ_k} by an induction using the multiplicity formula above. Furthermore, the trivial representation $\mathbb{1}$ appears in $V_{\Lambda_k}^{\otimes 2}$ so once V_λ appears in an odd (resp. even) tensor power of V_{Λ_k} it will appear in *every* odd (resp. even) tensor power thereafter. □.

The vector (or defining) representation of $\mathfrak{so}_{2k+1}$ has highest weight $\Lambda_1 = \varepsilon_1$. We will only need to know the decomposition for tensoring V_{Λ_1} with simple objects whose highest weight has integer entries:

Example 4.3. The weights of V_{Λ_1} are the zero weight together with $W(\Lambda_1) = \{\pm\varepsilon_i : 1 \leq i \leq k\}$. The decomposition algorithm as a $\mathfrak{so}_{2k+1}$-module is (for integral weights μ):

$$V_{\Lambda_1} \otimes V_\mu \cong \delta(\mu) V_\mu + \bigoplus_{W_1} V_{\mu + w(\Lambda_1)}$$

where $W_1 = \{w \in W : w(\Lambda_1) \in P_+\}$ and $\delta(\mu) = 1$ if $\langle \mu, \varepsilon_k \rangle > 0$ and zero otherwise. Since the dominant weights in $\overline{C_\ell} \setminus C_\ell$ all have integer entries and μ is distance at least 1 from the hyperplane spanned by $\overline{C_\ell} \setminus C_\ell$ we conclude that $P_+ \cap (\mu + W(\Lambda_1)) \subset \overline{C_\ell}$. Hence the W_ℓ-antisymmetrization has the effect of discarding those $V_{\mu+w(\Lambda_1)}$ with $\mu + \Lambda_1 \in \overline{C_\ell} \setminus C_\ell$. So for $\mu, \nu \in C_\ell \cap \mathbb{Z}^k$ we compute:

$$N_{\Lambda_1 \mu}^{\nu} = \begin{cases} 1 & \text{if } \nu = \mu \pm \varepsilon_i \text{ for some } 1 \leq i \leq k \\ 1 & \text{if } \mu = \nu \text{ and } \langle \mu, \varepsilon_k \rangle > 0 \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

As in Example 4.1, we can use this computation to conclude that V_{Λ_1} generates the subcategory of objects labeled by integer weights. It is slightly trickier to show that, in fact, every object labeled by an integer weight appears in both an even *and* an odd power of V_{Λ_1}. The trick is to find a μ with $|\mu| = s$ odd and $\langle \mu, \varepsilon_k \rangle > 0$. Then V_μ first appears in $V_{\Lambda_1}^s$ (that is, s is minimal with this property). But then V_μ appears in $V_{\Lambda_1}^{s+1}$ by the rule above. Since $s+1$ is even, by applying our rule again and using the fact that $N_{\Lambda_1 \mu}^{\nu}$ are completely symmetric we see that V_{Λ_1} appears in $V_{\Lambda_1}^{(s-1)(s+1)}$. Thus *every* object labeled by an integer weight appears in an even tensor power of V_{Λ_1}. By considering cases and applying this argument again we conclude that the same is true for odd tensor powers of V_{Λ_1}.

4.4. **Character Analysis.** Eventually we want to describe all (irreducible) characters of the ring $Gr(\mathcal{F})$. Our main source of characters are the *q-characters* of $Gr(Rep(\mathfrak{so}_{2k+1}))$, which are nothing more than ordinary characters of the ring $Gr(Rep(U_q\mathfrak{so}_{2k+1}))$ for q generic. To each $\nu \in Q$ (the root lattice) there is a certain "diagonalizable" element in the quantum group $U_q(\mathfrak{so}_{2k+1})$ denoted by H_ν on which the irreducible characters χ_λ are defined for any $\lambda \in P_+$. This is completely analogous to the classical situation where the characters act on a maximal abelian subalgebra.

$$\chi_\lambda(H_\nu) = \frac{1}{\delta_B(H_\nu)} \sum_{w\in W} \varepsilon(w) q^{\langle w(\lambda+\rho),\nu\rangle}$$

where

$$\delta_B(H_\nu) = \sum_{w\in W} \varepsilon(w) q^{\langle w(\rho),\nu\rangle}$$

is the *Weyl denominator*. Recall that $[n](q-q^{-1}) = q^n - q^{-n}$. An important computation due to Weyl [Wy] gives us the product form

$$\delta_B(H_\nu) = \prod_{\alpha\in\Phi_+} [\frac{1}{2}\langle\alpha,\nu\rangle]$$

(see [GWa] Chapter 7 for a more modern treatment). The $\frac{1}{2}$ appears here because we have normalized the form $\langle\ ,\ \rangle$ to be twice the form used in the classical theory. (Note that $\frac{1}{2}\langle\alpha,\nu\rangle$ is a integer since both α and ν have integer entries.)

For any fixed $\nu \in Q$ the characters χ_λ satisfy:

1. $\chi_{\mathbf{0}}(H_\nu) = 1$
2. $\chi_\lambda(H_\nu)\chi_\mu(H_\nu) = \sum_\kappa m^\kappa_{\lambda\mu}\chi_\kappa(H_\nu)$ where
$$m^\kappa_{\lambda\mu} = \dim \mathrm{Hom}_{U_q\mathfrak{so}_{2k+1}}(V_\kappa, V_\lambda \otimes V_\mu)$$

The first property is clear, while the second is a fundamental result in classical representation theory.

Now suppose q^2 is a primitive ℓth root of unity. Notice that if $\nu = 2\rho$ and $\lambda \in \overline{C_\ell}$ Weyl's formula give us:

$$\chi_\lambda(H_{2\rho}) = \dim_{\mathcal{F}}(V_\lambda).$$

This motivates the following notation:

Definition 4.4. Let $\mu \in P_+ \setminus \mathbb{Z}^k$ so that $\mu + \rho \in P_+ \cap Q$ (i.e. $p(\mu+\rho) = 1$). Then for all $\lambda \in P_+$ we define

$$\dim^\mu_{\mathcal{F}}(V_\lambda) := \chi_\lambda(H_{\mu+\rho}).$$

The following technical lemma gives the precise criterion for a character of $Gr(Rep(U_q(\mathfrak{so}_{2k+1})))$ to specialize to a character of $Gr(\mathcal{F})$:

Lemma 4.5. *The specialization of a character $\chi_\kappa(H_\nu)$ to $Gr(\mathcal{F})$ gives a character of $Gr(\mathcal{F})$ if and only if:*

3. *$\chi_\kappa(H_\nu) = \varepsilon(w)\chi_{w\cdot\kappa}(H_\nu)$ for all $\kappa \in C_\ell$, all $w \in W_\ell$ such that $w\cdot\kappa \in P_+$ and q^2 an ℓth root of unity, ℓ odd.*

Proof. Setting $W_\kappa = \{w \in W_\ell : w\cdot\kappa \in P_+\}$ for $\kappa \in \overline{C_\ell}$, the second property of characters χ_λ becomes:

$$\chi_\lambda(H_\nu)\chi_\mu(H_\nu) = \sum_i m^{\mu_i}_{\lambda\mu}\chi_{\mu_i}(H_\nu) =$$

$$\sum_{\kappa\in\overline{C_\ell}} \left(\sum_{w\in W_\kappa} \varepsilon(w) m^{w\cdot\kappa}_{\lambda\mu}\right) \chi_\kappa(H_\nu) = \sum_{\kappa\in C_\ell} N^\kappa_{\lambda\mu}\chi_\kappa(H_\nu)$$

since to every $\mu_i \in P_+$ there is a unique $\kappa \in \overline{C_\ell}$ so that $w \cdot \kappa = \mu_i$ for some $w \in W_\ell$ and $N_{\lambda\mu}^{\kappa} = 0$ if $\kappa \in \overline{C_\ell} \setminus C_\ell$. □

To prove Property 3 in the above lemma we need only verify it for simple reflections s_i, t_ℓ since they generate W_ℓ. Moreover, we need only consider the numerator of $\chi_\kappa(H_\nu)$ as the denominator $\delta_B(H_\nu)$ does not depend on κ. So the veracity of Property 3 will follow from the following lemma:

Lemma 4.6. $\sum_{w\in W} \varepsilon(w) q^{\langle w(r\cdot\kappa+\rho),\nu\rangle} = \varepsilon(r) \sum_{w\in W} \varepsilon(w) q^{\langle w(\kappa+\rho),\nu\rangle}$ *for r a simple reflection and $\nu \in Q$.*

Proof. Define $w' \in W$ by $w'(\lambda) = \lambda - \langle\lambda, \varepsilon_1\rangle\varepsilon_1$ and observe that $\varepsilon(w') = -1$ as w' just changes the sign of the first coordinate of λ. We compute:

$$\langle w(t_\ell \cdot \kappa + \rho), \nu\rangle = \langle t_\ell(\kappa + \rho) - \rho + \rho, w^{-1}(\nu)\rangle =$$
$$\langle(\kappa + \rho) - \langle\kappa + \rho, \varepsilon_1\rangle\varepsilon_1 + \ell\varepsilon_1, w^{-1}(\nu)\rangle = \langle ww'(\kappa + \rho), \nu\rangle + \ell\langle\varepsilon_1, \nu\rangle$$

Since $\ell\langle\varepsilon_1, \nu\rangle$ is an even multiple of ℓ and $\varepsilon(t_\ell) = -1$, we have:

$$\sum_{w\in W} \varepsilon(w) q^{\langle w(t_\ell\cdot\kappa+\rho),\nu\rangle} = \varepsilon(t_\ell) \sum_{w\in W} \varepsilon(w) q^{\langle w(\kappa+\rho),\nu\rangle}$$

after reindexing the sum. The computation for s_i is slightly less complicated, and just follows from the fact that $\chi_\kappa(H_\nu)$ is an antisymmetrization with respect to the Weyl group of the characters of the finite abelian group $\ell P/Q$. It can also be computed directly as for t_ℓ. Thus we have proved the lemma. □

Thus the specialization to roots of unity and restriction to C_ℓ of the characters $\chi_\kappa(H_\nu)$ are indeed characters of the ring $Gr(\mathcal{F})$.

Next we prove the following crucial:

Lemma 4.7. $\dim_{\mathcal{F}}^{\Lambda_k}(V_\lambda)$ *is positive for all $\lambda \in C_\ell$ for $q = e^{\pi i/\ell}$.*

Proof. First we consider the numerator

$$\sum_{w\in W} \varepsilon(w) q^{\langle w(\lambda+\rho),\Lambda_k+\rho\rangle}$$

of $\dim_{\mathcal{F}}^{\Lambda_k}(V_\lambda)$. Observe that the positive coroots $\check{\alpha} \in \check{\Phi}_+$ are $\frac{1}{2}$ the positive roots Φ_+^C of type C (corresponding to $\mathfrak{sp}_{2k}$). In the classical theory we would get exactly the positive roots of type C, but we are using twice the classical form. Furthermore $\Lambda_k + \rho = \rho'$ is one-half the sum of the positive roots of type C and is thus the sum of the positive coroots as we have defined them. Moreover, the Weyl group W is the same for these two algebras. Let $(\, , \,)$ be the usual inner product on Euclidean space, so that $2(a, b) = \langle a, b\rangle$. We have that

$$\sum_{w\in W} \varepsilon(w) q^{\langle w(\lambda+\rho),\Lambda_k+\rho\rangle} = \sum_{w\in W} \varepsilon(w) q^{\langle\lambda+\rho,w(\rho')\rangle} = \sum_{w\in W} \varepsilon(w) q^{(2(\lambda+\rho),w(\rho'))}$$
$$= \prod_{\beta\in\Phi_+^C} [(\lambda + \rho, \beta)] = \prod_{\check{\alpha}\in\check{\Phi}_+} [2(\lambda + \rho, \check{\alpha})] = \prod_{\check{\alpha}\in\check{\Phi}_+} [\langle\lambda + \rho, \check{\alpha}\rangle]$$

by the observations above and the classical Weyl denominator factorization for type C. The same computation for $\lambda = \mathbf{0}$ shows that the denominator of $\dim_{\mathcal{F}}^{\Lambda_k}(V_\lambda)$ also factors nicely so that when we evaluate at $q = e^{\pi i/\ell}$ we get:

$$\begin{aligned}\dim_{\mathcal{F}}^{\Lambda_k}(V_\lambda) &= \prod_{\check{\alpha}\in\check{\Phi}_+} \frac{[\langle\lambda+\rho,\check{\alpha}\rangle]}{[\langle\rho,\check{\alpha}\rangle]} = \prod_{\check{\alpha}\in\check{\Phi}_+} \frac{q^{\langle\lambda+\rho,\check{\alpha}\rangle} - q^{-\langle\lambda+\rho,\check{\alpha}\rangle}}{q^{\langle\rho,\check{\alpha}\rangle} - q^{-\langle\rho,\check{\alpha}\rangle}} \\ &= \prod_{\check{\alpha}\in\check{\Phi}_+} \frac{\sin(\langle\lambda+\rho,\check{\alpha}\rangle\pi i/\ell)}{\sin(\langle\rho,\check{\alpha}\rangle\pi i/\ell)}.\end{aligned}$$

Now we see that when $\lambda \in C_\ell$, $\langle\lambda+\rho,\check{\alpha}\rangle < \ell$ for all $\check{\alpha} \in \check{\Phi}_+$ so that each factor in the above product is positive. □

We end this section with an important uniqueness theorem which relies on the classical theorem of Perron and Frobenius found in [Ga]. Recall that a positive matrix is a matrix whose entries are all strictly positive.

Proposition 4.8 (Perron-Frobenius). *A positive matrix A always has a positive real eigenvalue of multiplicity one whose modulus exceeds the moduli of all other eigenvalues. Furthermore the corresponding eigenvector may be chosen to have only positive real entries and is the unique eigenvector with that property.*

We now proceed to prove:

Theorem 4.9. *Evaluating $\dim_{\mathcal{F}}^{\Lambda_k}(V_\lambda)$ at $e^{\pi i/\ell}$ gives the only character of $Gr(\mathcal{F})$ that is positive for all $\lambda \in C_\ell$.*

Proof. We observed in 2.3 that if $f : C_\ell \to \mathbb{C}$ is a character of $Gr(\mathcal{F})$ then the vector $\mathbf{f} = (f(\lambda))_{\lambda\in C_\ell}$ must be a simultaneous eigenvector of the set $\mathcal{M} := \{N_\lambda\}$, $\lambda \in C_\ell$. In fact, using the definition of N_λ one computes that $N_\lambda(\mathbf{f}) = f(\lambda)\mathbf{f}$. So if we can show that $N_{\Lambda_k} \in \mathcal{M}$ has only one positive eigenvector we will have proved the theorem. In the proof of Lemma 4.2 we saw that for some odd integer s, the matrix $N_{\Lambda_k}^s + N_{\Lambda_k}^{s+1}$ has all positive entries. So one may apply the Perron-Frobenius Theorem to the matrix $N_{\Lambda_k}^s + N_{\Lambda_k}^{s+1}$ to see that it has a *unique* positive eigenvector. But N_{Λ_k} is a (symmetric) diagonalizable matrix, so it has the same eigenvectors as $N_{\Lambda_k}^s + N_{\Lambda_k}^{s+1}$. Since $\dim_{\mathcal{F}}^{\Lambda_k}(V_\lambda)$ at $e^{\pi i/\ell}$ was shown to be positive in Lemma 4.7, we are done. □

4.5. **The Involution.** Next we define an involution ϕ of C_ℓ that will be central to the analysis of the characters of $Gr(\mathcal{F})$. Let $\gamma \in C_\ell$ be such that $|\gamma|$ is maximal, explicitly, $\gamma = (\frac{\ell-2k}{2}, \dots, \frac{\ell-2k}{2})$. Further denote by w_1 the element of the Weyl group W such that $w_1(\mu_1, \dots, \mu_k) = (\mu_k, \dots, \mu_1)$. Define $\phi(\lambda) := \gamma - w_1(\lambda)$. It is clear that ϕ is a bijective map from C_ℓ to itself and that $\phi^2(\lambda) = \lambda$, and that $\phi \notin W_\ell$ as no $\lambda \in P_+$ is fixed by ϕ. The following lemma describes the key property of ϕ.

Lemma 4.10. *For q^2 a primitive ℓth root of unity the involution ϕ preserves $|\dim_{\mathcal{F}}^\mu|$ (for $\mu \in P_+ \setminus \mathbb{Z}^k$), that is,*

$$\dim_{\mathcal{F}}^\mu(V_\lambda) = \pm \dim_{\mathcal{F}}^\mu(V_{\phi(\lambda)}) \tag{16}$$

In particular (by setting $\mu = \rho$) this holds for the categorical dimension $\dim_{\mathcal{F}}$ of $\mathcal{F}$.

Proof. Fix $\mu \in P_+ \setminus \mathbb{Z}^k$ and a choice of a primitive ℓth root of unity q^2 (so $q^\ell = \pm 1$). First consider $\sum_{w\in W} \varepsilon(w) q^{\langle \lambda+\rho, w(\mu+\rho)\rangle}$ the numerator of $\dim^\mu_{\mathcal{F}}(V_\lambda)$. We compute

$$\begin{aligned}\langle \phi(\lambda)+\rho, w(\mu+\rho)\rangle &= \langle \gamma - w_1(\lambda)+\rho, w(\mu+\rho)\rangle \\ &= \langle w_1(\gamma-\lambda+\rho+w_1(\rho)-\rho), w(\mu+\rho)\rangle \\ &= \langle \gamma+\rho+w_1(\rho), w_1 w(\mu+\rho)\rangle + \langle \lambda+\rho, -w_1 w(\mu+\rho)\rangle \\ &= \ell\cdot \sum_i (w_1 w(\mu+\rho))_i + \langle \lambda+\rho, -w_1 w(\mu+\rho)\rangle.\end{aligned}$$

Now $t(\mu) := \sum_i (w_1 w(\mu+\rho))_i = \sum_i (w(\mu+\rho))_i$ is an integer whose parity is the same as that of $\sum_i (\mu+\rho)_i$ and depends only on μ (and the rank k), and $q^\ell = \pm 1$ so $q^{\ell\cdot t(\mu)} = \pm 1$ and we have

$$\begin{aligned}&\sum_{w\in W} \varepsilon(w) q^{\langle \phi(\lambda)+\rho, w(\mu+\rho)\rangle} = \sum_{w\in W} \pm\varepsilon(w) q^{\langle \lambda+\rho, -w_1 w(\mu+\rho)\rangle} \\ &= \pm \sum_{w'\in W} \varepsilon(w') q^{\langle \lambda+\rho, w'(\mu+\rho)\rangle}\end{aligned}$$

where $w' = -w_1 w$. Since the denominator of $\dim^\mu_{\mathcal{F}}(V_\lambda)$ is independent of λ the lemma is true for $\mu \in P_+ \cap \frac{1}{2}\mathbb{Z}^k \setminus \mathbb{Z}^k$. □

Let us pause for a moment to nail down exactly which sign $\dim^\rho_{\mathcal{F}}(V_{\phi(\lambda)}) = \dim_{\mathcal{F}}(V_{\phi(\lambda)})$ has in terms of $\dim^\rho_{\mathcal{F}}(V_\lambda)$. Here there are two factors governing signs of the characters: $\varepsilon(-w_1)$ and the parity of $\sum_i w(2\rho)_i$. One has that:

$$\varepsilon(-w_1) = \begin{cases} (-1)^{k/2} & \text{for } k \text{ even} \\ (-1)^{(k-1)/2} & \text{for } k \text{ odd}\end{cases}$$

Furthermore we compute: $q^{l\sum_i w(2\rho)_i} = (q^\ell)^k$ so we have the following result:

Scholium 4.11. *If $q^\ell = -1$ then*

$$\dim_{\mathcal{F}}(V_{\phi(\lambda)}) = \begin{cases} \dim_{\mathcal{F}}(V_\lambda) & k \equiv 0 \bmod 4 \\ \dim_{\mathcal{F}}(V_\lambda) & k \equiv 1 \bmod 4 \\ -\dim_{\mathcal{F}}(V_\lambda) & k \equiv 2 \bmod 4 \\ -\dim_{\mathcal{F}}(V_\lambda) & k \equiv 3 \bmod 4\end{cases}$$

Whereas if $q^\ell = 1$:

$$\dim_{\mathcal{F}}(V_{\phi(\lambda)}) = \begin{cases} \dim_{\mathcal{F}}(V_\lambda) & k \equiv 0 \bmod 4 \\ -\dim_{\mathcal{F}}(V_\lambda) & k \equiv 1 \bmod 4 \\ -\dim_{\mathcal{F}}(V_\lambda) & k \equiv 2 \bmod 4 \\ \dim_{\mathcal{F}}(V_\lambda) & k \equiv 3 \bmod 4\end{cases}$$

The following important lemma gives the decomposition rule for tensoring with the object in $\mathcal{F}$ labeled by γ.

Lemma 4.12. *$V_\gamma \otimes V_\mu = V_{\phi(\mu)}$ for all $\mu \in C_\ell$.*

Proof. By Lemmas 4.10 and 4.7 we know that $\dim^{\Lambda_k}_{\mathcal{F}}(V_\gamma) = \dim^{\Lambda_k}_{\mathcal{F}}(\mathbb{1}) = 1$ since $\phi(\mathbf{0}) = \gamma$. So

$$\dim^{\Lambda_k}_{\mathcal{F}}(V_\gamma \otimes V_\mu) = \dim^{\Lambda_k}_{\mathcal{F}}(V_\mu) = \dim^{\Lambda_k}_{\mathcal{F}}(V_{\phi(\mu)}).$$

Recall from 3.4 that

$$\dim \mathrm{Hom}_{\mathcal{F}}(V_\gamma \otimes V_\mu, V_\nu) = N_{\gamma\mu}^{\nu} = \sum_{W_\nu} \varepsilon(w) m_{\gamma\mu}^{w\cdot\nu}$$

where $W_\nu = \{w \in W_l : w \cdot \nu \in P_+\}$ and

$$m_{\gamma\mu}^{w\cdot\nu} = \dim \mathrm{Hom}_{U_q\mathfrak{so}_{2k+1}}(V_\gamma \otimes V_\mu, V_{w\cdot\nu}).$$

Observe that the weight $\phi(\mu) = \gamma - w_1(\mu)$ is in C_ℓ and $m_{\gamma\mu}^{\phi(\mu)} = 1$ (see 4.2). The only way that $V_\phi(\mu)$ might fail to appear in the $\mathcal{F}$ decomposition is if $\phi(\mu)$ were equal to a reflection (under the dot action of W_ℓ) of $\gamma + \kappa$ for some $\kappa \in P(\mu)$ (notice this also covers weights in other Weyl chambers). To see that this is impossible, we use a geometric argument, although it is really nothing more than an adaptation of the classical outer multiplicity formula. First note that γ is a positive distance from all walls of reflection under the dot action of W_ℓ. Next observe that the straight line segment from γ to $\gamma + \kappa$ has Euclidean length $|\kappa| \leq |\mu|$. So the reflected piecewise linear path from γ to $w \cdot (\gamma + \kappa)$ will not be straight, and will have total length $|\kappa|$ as well. Thus the straight line segment from γ to $w \cdot (\gamma + \kappa)$ must have length strictly less than $|\mu|$, whereas the straight line segment from γ to $\phi(\mu)$ has length $|\mu|$. So $V_\phi(\mu)$ appears in the $\mathcal{F}$ decomposition of $V_\gamma \otimes V_\mu$. But since $\dim_{\mathcal{F}}^{\Lambda_k}$ is positive on C_ℓ and

$$\dim_{\mathcal{F}}^{\Lambda_k}(V_{\phi(\mu)}) = \dim_{\mathcal{F}}^{\Lambda_k}(V_\gamma \otimes V_\mu) = \sum_{\nu} N_{\gamma\mu}^{\nu} \dim_{\mathcal{F}}^{\Lambda_k}(V_\nu)$$

it is clear that $V_{\phi(\mu)}$ is the *only* object that appears in the decomposition. □

Remark 4.13. This result can also be derived from [LT], Remark 3.9. Le and Turaev studied symmetries in more general settings for topological applications. The involution ϕ also appears in slightly more general setting in the paper [S] by S. Sawin.

4.6. **The Family $\mathfrak{F}$ Summarized.** Let us collect together the important facts mentioned so far:

1. For a fixed k and ℓ the corresponding family $\mathfrak{F}$ of categories has a common Grothendieck semiring, denoted $Gr(\mathcal{F})$.
2. $Gr(\mathcal{F})$ has a unique positive character.
3. The involution ϕ preserves characters of $Gr(\mathcal{F})$ up to a sign, and is induced by tensoring with V_γ.
4. $Gr(\mathcal{F})$ has at most $|C_\ell|$ distinct characters each of which is a simultaneous eigenvector of the set $\mathcal{M}$ of matrices.

5. Braid Group Representations

In this section we analyze the representations $\mathbb{C}\mathcal{B}_n \to \mathrm{End}_{\mathcal{F}}(V^{\otimes n})$ with an eye towards realizing these centralizer algebras as (specializations of) quotients of BMW-algebras $C_f(r, q)$ which we will define below.

Recall from Example 4.3 that every object labeled by an integer weight appears in an *even* and an *odd* tensor power of the object V_{Λ_1}. Introduce the object $V = V_{\Lambda_1} \otimes V_\gamma \cong V_{\phi(\Lambda_1)}$. By Lemma 4.12 we see that V is a generator for the category $\mathcal{F}$, since $V_\gamma^{\otimes 2} \cong \mathbb{1}$. We saw before that V_{Λ_k} was also a generator, but V has the advantage that $V^{\otimes 2}$ *always* decomposes as the direct sum of 3 simple objects regardless of the rank k. So we can take advantage of a computation of Tuba and Wenzl ([TuW2] proof of Lemma 6.3):

Lemma 5.1. *Suppose $\mathcal{O}$ is a semisimple ribbon category generated by a simple object X and $X^{\otimes 2} \cong \mathbb{1} \oplus Y \oplus Z$ with Y and Z simple objects. If the eigenvalues of $c_{X,X}$ are c_1, c_2 and c_3 respectively on Y, Z and $\mathbb{1}$ then:*

$$\dim_{\mathcal{O}}(X) = \pm \frac{c_3^2 + c_1c_2 - c_3(c_1 + c_2)}{c_3(c_1^{-1} + c_2^{-1})}.$$

The proof of this lemma relies upon Lemma 2.4 and the explicit computations in [TuW1].

5.1. **BMW-Algebras.** The algebras $C(r, q)$ are quotients of the group algebra of Artin's braid group $\mathcal{B}_f$ and were studied extensively in [W1] and [TW2], and more recently in [TuW2].

Definition 5.2. Let $r, q \in \mathbb{C}$ and $f \in \mathbb{N}$, then $C_f(r, q)$ is the $\mathbb{C}$-algebra with invertible generators $g_1, g_2, \ldots, g_{f-1}$ and relations:

(B1) $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$,
(B2) $g_ig_j = g_jg_i$ if $|i - j| \geq 2$,
(R1) $e_ig_i = r^{-1}e_i$,
(R2) $e_ig_{i-1}^{\pm 1}e_i = r^{\pm 1}e_i$,
where e_i is defined by
(E1) $(q - q^{-1})(1 - e_i) = g_i - g_i^{-1}$

Notice that (E1) and (R1) imply

$$(g_i - r^{-1})(g_i - q)(g_i + q^{-1}) = 0$$

for all i. So the image of g_i on any finite dimensional representation has at most three eigenvalues, which are distinct if $q^2 \neq -1$ and $r \neq \pm q^{\pm 1}$. Notice further that the image of e_i is a multiple of the projection onto the g_i-eigenspace corresponding to eigenvalue r^{-1}.

There exists a trace tr on the family of algebras $C_f(r, q)$ uniquely determined by the values on the generators, and inductively defined by (see [W1]):

(T1) $tr(1) = 1$
(T2) $tr(g_i) = r(\frac{q-q^{-1}}{r-r^{-1}+q-q^{-1}})$,
(T3) $tr(axb) = tr(ab)tr(x)$ for $a, b \in C_{f-1}(r, q)$ and $x \in \{g_{f-1}, e_{f-1}, 1\}$.

The existence of such a trace comes from the well-known Kauffmann link invariant. When q is a root of unity and r is plus or minus a power of q then the algebras $E_f := C_f(r, q)/Ann(tr)$ are finite dimensional and semisimple and hence isomorphic to a direct sum of full matrix algebras.

In [TW2] the authors construct a family of self-dual Hermitian ribbon categories from the sequence of algebras $\mathbb{C} \subset \cdots E_f \subset E_{f+1} \subset \cdots$ for various choices of r and q. The objects in these categories are the idempotents in the algebras E_f, $f \geq 1$ and the morphisms are images of *tangles*. Since the algebra E_f is a quotient of the group algebra of the braid group $\mathbb{C}B_f$, the braiding in the category is obtained directly as images of elements in E_f. The construction is quite involved, so we will be content to outline the important properties leaving the interested reader to seek details in the above reference as well as [TuW2].

5.2. **The BC-Case and the Family $\mathfrak{V}$.** Fix q with q^2 a primitive ℓth root of unity, and let $r = -q^{2k}$. Denote by $\mathcal{V}$ the corresponding self-dual Hermitian ribbon category as constructed in [TW2]. This is known as the *ortho-symplectic* or BC-case in the literature. For ℓ odd, we have the following:

1. The simple objects of $\mathcal{V}$ are labeled by Ferrer's diagrams $\lambda \in \Gamma(k,\ell)$ where:
$$\Gamma(k,\ell) := \{\lambda : \lambda_1' + \lambda_2' \leq 2k+1, \lambda_1 \leq (\ell - 2k - 1)/2\}$$
with λ_i (resp. λ_i') the number of boxes in the ith row (resp. column) of λ (see [W1]).
2. The object $X := X_{\square}$ generates $\mathcal{V}$ (see [W1]).
3. For $\mu \in \Gamma(k,\ell)$, X_μ is a simple subobject of the tensor product $X \otimes X_\lambda$ if and only if μ can be obtained by adding/deleting one box to/from λ (see [TW2]).
4. $Gr(\mathcal{V}) \cong Gr(Rep(O(2k+1)))/\mathcal{J}$ where $\mathcal{J}$ is some ideal. (see [TuW2]).
5. $\dim_{\mathcal{V}}(X) = \frac{[-2k]}{[1]} + 1$ (see [W1]).
6. The eigenvalues of the braiding morphism $c_{X,X}$ on the simple subobjects $\{\mathrm{Id}, X_{[2]}, X_{[1^2]}\}$ are respectively either $\{-q^{-2k}, q, -q^{-1}\}$ or $\{-iq^{-2k}, iq, -iq^{-1}\}$. (depending on a choice of a braiding, see [TuW2]).
7. By replacing the braiding morphism $c_{X,X}$ by its negative, inverse or negative-inverse we get 3 new inequivalent ribbon categories with the same Grothendieck semiring as $\mathcal{V}$.
8. There is an algebra isomorphism $\mathrm{End}_{\mathcal{V}}(X^{\otimes n}) \cong E_n$ that preserves the $\mathbb{C}\mathcal{B}_n$-module structure (see [TuW2]).

The key theorem we will use is the following special case of the main result in [TuW2] (Theorem 9.5):

Proposition 5.3. *Fix k and ℓ. Let $\mathfrak{V}$ be the family of braided tensor categories constructed from BMW-algebras where q^2 is any primitive ℓth root of unity and $r = -q^{2k}$, and $c_{X,X}$ is one of the four braiding morphisms as in item 7 above. Then any braided tensor category $\mathcal{O}$ with $Gr(\mathcal{O}) \cong Gr(\mathcal{V})$ such that the braiding morphism $c_{Y,Y}$ where $Y \in \mathcal{O}$ is the object corresponding to $X \in \mathcal{V}$ has 3 distinct eigenvalues is equivalent (as a braided tensor category) to a member of the family $\mathfrak{V}$.*

6. Main Theorem

We now proceed to prove:

Theorem 6.1. *Fix $\mathcal{F} \in \mathfrak{F}$. Then as a braided tensor category, $\mathcal{F}$ is equivalent to some $\mathcal{V} \in \mathfrak{V}$.*

The proof is outlined as follows:

Step 0 It is sufficient to show the theorem for any fixed $\mathcal{F} \in \mathfrak{F}$.
Step 1 The image of $\mathbb{C}\mathcal{B}_n$ in $\mathrm{End}_{\mathcal{F}}(V^{\otimes n})$ is a quotient of E_n.
Step 2 There exists a $\mathcal{V} \in \mathfrak{V}$ such that as algebras $\mathrm{End}_{\mathcal{V}}(X^{\otimes n}) \cong \mathrm{End}_{\mathcal{F}}(V^{\otimes n})$.
Step 3 $Gr(\mathcal{V}) \cong Gr(\mathcal{F})$.
Step 4 As braided tensor categories $\mathcal{V}$ and $\mathcal{F}$ are equivalent up to 4 possible choices of braiding morphism $c_{V,V}$.

Remark 6.2. It should be emphasized that the fusion rules of $\mathfrak{F}$ are *a priori* only obtained as a quotient of the representation category of $O(2k+1)$ which is the "integer half" of the representation category of $\mathfrak{so}_{2k+1}$ (i.e. lacking the spinnor representations). So while it was well-known that there is a relationship between $\mathfrak{V}$ and $\mathfrak{F}$ (see [W1]), we show that *all* of $\mathfrak{F}$ can be obtained as a quotient of the Turaev-Wenzl category.

6.1. **Proof of Step 0.** Since all $\mathcal{F} \in \mathfrak{F}$ share the same Grothendieck semiring and the eigenvalues are distinct by Proposition 3.2 (we explicitly compute them below), 5.3 implies that once we have established the theorem for some $\mathcal{F} \in \mathfrak{F}$ we will be done.

6.2. **Proof of Step 1.** Step 1 will follow as soon as we show that the images of the braid generators: $R_i := \mathrm{Id}_V^{\otimes(i-1)} \otimes c_{V,V} \otimes \mathrm{Id}_V^{\otimes(n-i-1)}$ satisfy the defining relations of $C_n(r, q)$ as well as the trace conditions (see 5.2) for appropriate choice of r and q.

The object $V^{\otimes 2}$ decomposes as the sum of the three objects $\mathbb{1}$, $V_1 := V_{(2,0,\ldots,0)}$ and $V_2 := V_{(1,1,0,\ldots,0)}$. Applying Proposition 3.2 we see that the eigenvalues of $(c_{V,V})^2$ on V_1, V_2 and $\mathbb{1}$ depend on the parity of k and the sign of $q^\ell = \pm 1$ as follows:

1. On V_1:

$$c_1^2 = \begin{cases} -q^{-4} & \text{if } k \text{ odd and } q^\ell = -1 \\ q^{-4} & \text{otherwise} \end{cases} \tag{17}$$

2. On V_2:

$$c_2^2 = \begin{cases} -q^{4} & \text{if } k \text{ odd and } q^\ell = -1 \\ q^{4} & \text{otherwise} \end{cases} \tag{18}$$

3. On $\mathbb{1}$:

$$c_3^2 = \begin{cases} -q^{-8k} & \text{if } k \text{ odd and } q^\ell = -1 \\ q^{-8k} & \text{otherwise} \end{cases} \tag{19}$$

So the eigenvalues are $\{c_1, c_2, c_3\}$ either $\{\pm q^{-2}, \pm q^2, \pm q^{-4k}\}$ or $\{\pm iq^{-2}, \pm iq^2, \pm iq^{-4k}\}$ where the sign choices are independent. For simplicity (by Step 0) we assume that $q^\ell = -1$.

By Lemma 4.10 we have that $\dim_{\mathcal{F}}(V) = \pm \dim_{\mathcal{F}}(V_{\Lambda_1})$ and an easy computation using the equation for $\dim_{\mathcal{F}}$ in 3.2.9 we get:

$$\pm \dim_{\mathcal{F}}(V) = \frac{[4k]}{[2]} + 1 \tag{20}$$

Next we make a change of parameter: $\tilde{q} \to -q^2$. Observe that $\tilde{q}$ is still a primitive ℓth root of unity with $\tilde{q}^\ell = -1$, so by Step 0 we can proceed with this altered category. This change gives us:

$$\pm \dim_{\mathcal{F}}(V) = \frac{-[2k]_{\tilde{q}}}{[1]_{\tilde{q}}} + 1 \tag{21}$$

and $\{c_1, c_2, c_3\}$ is either $\{\pm\tilde{q}^{-1}, \pm\tilde{q}, \pm\tilde{q}^{-2k}\}$ or $\{\pm i\tilde{q}^{-1}, \pm i\tilde{q}, \pm i\tilde{q}^{-2k}\}$.

Using Lemma 5.1 we can test the possible choices of $\{c_1, c_2, c_3\}$ by computing $\pm \dim_{\mathcal{F}}(V)$ and comparing with $\frac{-[2k]_{\tilde{q}}}{[1]_{\tilde{q}}} + 1$. A (somewhat tedious) computation forces $\{c_1, c_2, c_3\}$ to be one of the two choices $\pm\{-\tilde{q}^{-1}, \tilde{q}, -\tilde{q}^{-2k}\}$ for k even and $\pm\{-i\tilde{q}^{-1}, i\tilde{q}, -i\tilde{q}^{-2k}\}$ for k odd. Now by changing the sign of $c_{X,X}$, we can change the sign of the corresponding eigenvalues for the target category $\mathcal{V}$, so we assume the eigenvalues $\{c_1, c_2, c_3\}$ are $\{-\tilde{q}^{-1}, \tilde{q}, -\tilde{q}^{-2k}\}$ for k even and $\{-i\tilde{q}^{-1}, i\tilde{q}, -i\tilde{q}^{-2k}\}$ for k odd. So comparing with the defining relations for the BMW-algebras and setting $-\tilde{q}^{2k} = r$ we need to show (for the k even case):

(B1) $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$,
(B2) $R_i R_j = R_j R_i$ if $|i - j| \geq 2$,
(R1) $S_i R_i = r^{-1} S_i$,
(R2) $S_i R_{i-1}^{\pm 1} S_i = r^{\pm 1} S_i$,
where S_i is defined by
(E1) $(\tilde{q} - \tilde{q}^{-1})(\mathrm{Id} - S_i) = R_i - R_i^{-1}$
(T1) $tr_{\mathcal{F}}(\mathrm{Id}_V) = 1$

Table 1. Tensor Categories

Category	Labeling Set	Objects
$Rep(O(2k+1))$	Diagrams λ, $\lambda_1'+\lambda_2'\leq 2k+1$	W_λ
$\mathcal{V}$	$\Gamma(k,\ell)$	X_λ
$Rep(U_q\mathfrak{so}_{2k+1})$, $\lvert q\rvert\neq 1$	P_+	V_λ
$\mathcal{F}$	C_ℓ	V_λ

(T2) $tr_{\mathcal{F}}(R_i)=r(\frac{\tilde{q}-\tilde{q}^{-1}}{r-r^{-1}+\tilde{q}-\tilde{q}^{-1}})$,

(T3) $tr_{\mathcal{F}}((a\otimes \mathrm{Id}_V)x(b\otimes \mathrm{Id}_V)) = tr_{\mathcal{F}}((a\circ b)\otimes \mathrm{Id}_V)tr_{\mathcal{F}}(x)$ for $a,b\in \mathrm{End}_{\mathcal{F}}(V^{\otimes(f-1)})$ and $x\in \{R_{f-1},S_{f-1},1\}\subset \mathrm{End}(V^{\otimes f})$.

Here $tr_{\mathcal{F}}(a):=Tr_{\mathcal{F}}(a)/\dim_{\mathcal{F}}(V^{\otimes n})$ where $a\in \mathrm{End}_{\mathcal{F}}(V^{\otimes n})$. Relations (B1) and (B2) are immediate from the braiding axioms, and (T1) follows from the definition of the normalized trace $tr_{\mathcal{F}}$. Relation (R1) follows from the computation of the eigenvalues of R_i and definition (E1). To verify (R2) it is sufficient to consider $i=2$ and verify the relation on $\mathrm{End}_{\mathcal{F}}(V^{\otimes 3})$. Since $S_1\in \mathrm{End}_{\mathcal{F}}(V^{\otimes 2})$ is $1+\frac{r-r^{-1}}{\tilde{q}-\tilde{q}^{-1}}$ times the projection onto the subobject $\mathbb{1}$ in $V^{\otimes 2}$, we can apply Lemma 2.4 to get:

$$S_2R_1S_2=tr_{\mathcal{F}}(c_{V,V})(1+\frac{r-r^{-1}}{\tilde{q}-\tilde{q}^{-1}})S_2$$

so (R2) will follow from (T2). Applying 2.9 to the eigenspace decomposition $R_1=r^{-1}p_{\mathbf{1}}-\tilde{q}^{-1}p_{V_1}+\tilde{q}p_{V_2}$ acting on $V\otimes V$ and computing $\dim_{\mathcal{F}}(V_1)$ and $\dim_{\mathcal{F}}(V_2)$ from the definition it is a matter of simple algebra to verify (T2). All that remains is to verify (T3). But since the algebras $\mathrm{End}(V^{\otimes f})$ are semisimple and finite dimensional it is enough to show (T3) for a,b minimal idempotents. But this reduces to Lemma 2.5.

So we conclude that $\mathrm{End}(V^{\otimes n})$ contains a quotient of E_n for all n. □

6.3. **Proof of Step 2.** Using the fact that $E_n\cong \mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ for all n together with Step 1 we need only show that $\dim \mathrm{End}_{\mathcal{F}}(V^{\otimes n})=\dim \mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ for *any* $\mathcal{V}$ and $\mathcal{F}$ in their respective families to conclude that the action of $\mathbb{C}\mathcal{B}_n$ on these two algebras is the same.

Several tensor categories will be bandied about in what follows. Recall first the following sets:

(1) $\Gamma(k,\ell)=\{\lambda:\lambda_1'+\lambda_2'\leq 2k+1,\lambda_1\leq(\ell-2k-1)/2\}$. Here λ is a Ferrer's diagram, and λ_i' is the number of boxes in the ith column.

(2) $P_+=\{\lambda\in\mathbb{Z}^k\cup(\mathbb{Z}^k+\frac{1}{2}(1,1,\ldots,1)):\lambda_1\geq\lambda_2\geq\ldots\lambda_k\geq 0\}$

(3) $C_\ell=\{\lambda\in P_+:\frac{\ell-2k}{2}\geq\lambda_1\}$.

Table 1 will serve as a lexicon of notation. The first column is the category, the second the labeling set for simple objects, and the third the notation used for the simple object labeled by λ.

Next we note a few homomorphisms that exist between the Grothendieck semirings of these tensor categories.

1. As we mentioned 5.2, the ring $Gr(\mathcal{V})$ is a quotient of $Gr(Rep(O(2k+1)))$. Provided $\mu\in\Gamma(k,\ell)$ we have:
$$\dim \mathrm{Hom}_{\mathcal{V}}(X_\mu,X_\square\otimes X_\lambda)=\dim \mathrm{Hom}_{O(2k+1)}(W_\mu,W_\square\otimes W_\lambda).$$
2. Define a map from the set of $O(2k+1)$ dominant weights (Ferrer's diagrams with at most $2k+1$ boxes in the first two columns) to the integer weights of $\mathfrak{so}_{2k+1}$ by restricting and

differentiating the irreducible representations. Explicitly this associates to λ the Ferrer's diagram $\overline{\lambda}$ identical to λ except the first column has $\min\{2k+1-\lambda_1', \lambda_1'\}$ boxes (here λ_1' is the number of boxes in the first column of λ). By filling in zeros for empty rows, we express $\overline{\lambda}$ as a k-tuple in our standard notation for dominant weights of $\mathfrak{so}_{2k+1}$. The map $\lambda \to \overline{\lambda}$ induces a homomorphism from $Gr(Rep(O(2k+1)))$ to $Gr(Rep(\mathfrak{so}_{2k+1}))$. From this we deduce:

$$\dim \mathrm{Hom}_{\mathcal{V}}(W_\mu, W_\square \otimes W_\lambda) = \dim \mathrm{Hom}_{\mathfrak{so}_{2k+1}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\overline{\lambda}}).$$

3. For generic q, the semirings $Gr(Rep(\mathfrak{so}_{2k+1}))$ and $Gr(Rep(U_q\mathfrak{so}_{2k+1}))$ are isomorphic. For this reason we denote the simple objects from both categories by V_λ.
4. The category $\mathcal{F}$ is obtained from $Rep(U_q\mathfrak{so}_{2k+1})$ as a quotient. Heedless of any potential confusion, we denote the simple objects in $\mathcal{F}$ by V_λ as well. Recall from Example 4.3 that for any integer weight $\lambda \in C_\ell$:

$$V_\mu \subset V_{\Lambda_1} \otimes V_\lambda \iff V_\mu \subset V_{\Lambda_1} \otimes V_\lambda,\ \mu \in C_\ell.$$

Define a bijection $\Psi : \Gamma(k, \ell) \to C_\ell$ by

$$\Psi(\lambda) = \begin{cases} \overline{\lambda}, & \text{if } |\lambda| \text{ is even,} \\ \phi(\overline{\lambda}), & \text{if } |\lambda| \text{ is odd.} \end{cases} \tag{22}$$

Observing that the tensor product of any simple object in $\mathcal{V}$ (resp. $\mathcal{F}$) with the generating object X (resp. V) is multiplicity free, the algebras $\mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ and $\mathrm{End}_{\mathcal{F}}(V^{\otimes n})$ are isomorphic once we prove:

Lemma 6.3. *Let $\mu, \lambda \in \Gamma(k, \ell)$. Then*

$$\dim \mathrm{Hom}_{\mathcal{V}}(X_\mu, X \otimes X_\lambda) = \dim \mathrm{Hom}_{\mathcal{F}}(V_{\Psi(\mu)}, V \otimes V_{\Psi(\lambda)}).$$

Proof. Using the first homomorphism of Grothendieck semirings above and the assumption that $\mu \in \Gamma(k, \ell)$, we have

$$\dim \mathrm{Hom}_{\mathcal{V}}(X_\mu, X \otimes X_\lambda) = \dim \mathrm{Hom}_{O(2k+1)}(W_\mu, W_\square \otimes W_\lambda).$$

Restricting to $SO(2k+1)$, differentiating and applying the third homomorphism above we have

$$\dim \mathrm{Hom}_{U_q\mathfrak{so}_{2k+1}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\overline{\lambda}}) = \dim \mathrm{Hom}_{O(2k+1)}(W_\mu, W_\square \otimes W_\lambda).$$

Now we split into the two cases from the definition of Ψ:
Case I: $|\lambda|$ is even (so $|\mu|$ is odd)
Since $\overline{\mu} \in C_\ell$ and $V_{\Psi(\lambda)} = V_{\overline{\lambda}}$ we see that

$$\dim \mathrm{Hom}_{\mathcal{F}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\Psi(\lambda)}) = \dim \mathrm{Hom}_{U_q\mathfrak{so}_{2k+1}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\overline{\lambda}}) \tag{23}$$

Lemma 4.12 implies that $V_\gamma \otimes V_{\overline{\mu}} = V_{\phi(\overline{\mu})} = V_{\Psi(\mu)}$ as objects in $\mathcal{F}$, and similarly $V_\gamma \otimes V_{La_1} = V$. So tensoring with V_γ (see example 23) we have:

$$\dim \mathrm{Hom}_{\mathcal{F}}(V_{\Psi(\mu)}, V \otimes V_{\Psi(\lambda)}) = \dim \mathrm{Hom}_{\mathcal{F}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\Psi(\lambda)}).$$

Case II: $|\lambda|$ is odd (so $|\mu|$ is even)
In this case $V_{\Psi(\lambda)} = V_\gamma \otimes V_{\overline{\lambda}}$ and $V_{\Psi(\mu)} = V_{\overline{\mu}}$ so using the fact that $V_\gamma \otimes V_\gamma = \mathbb{1}$ we derive similarly that

$$\dim \mathrm{Hom}_{\mathcal{F}}(V_{\Psi(\mu)}, V \otimes V_{\Psi(\lambda)}) = \dim \mathrm{Hom}_{\mathcal{F}}(V_{\overline{\mu}}, V_{\Lambda_1} \otimes V_{\Psi(\lambda)})$$

in this case. □

This lemma implies that

$$\dim \mathrm{Hom}_{\mathcal{V}}(X_\nu, X^{\otimes n}) = \dim \mathrm{Hom}_{\mathcal{F}}(V_{\Psi(\nu)}, V^{\otimes n})$$

by an easy induction argument. Thus we have shown that $\dim \mathrm{End}_{\mathcal{F}}(V^{\otimes n}) = \dim \mathrm{End}_{\mathcal{V}}(X^{\otimes n})$. □

6.4. **Proof of Step 3.** In Step 1 we established that the action of $\mathbb{C}\mathcal{B}_n$ on $\mathrm{End}_{\mathcal{F}}(V^{\otimes n})$ factors through E_n which is isomorphic to $\mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ both as algebras and $\mathbb{C}\mathcal{B}_n$-modules. Combining this with Step 2 we conclude that $\mathrm{End}_{\mathcal{F}}(V^{\otimes n})$ and $\mathrm{End}_{\mathcal{V}}(X^{\otimes n})$ are isomorphic both as algebras and as $\mathbb{C}\mathcal{B}_n$-modules. Since V and X generate their respective categories, this implies that the Grothendieck semirings $Gr(\mathcal{F})$ and $Gr(\mathcal{V})$ are isomorphic. In fact, it is tedious but straightforward to show that:

Corollary 6.4. *Ψ induces an isomorphism $Gr(\mathcal{V}) \cong Gr(\mathcal{F})$.*

That is, the map Ψ defined above on the labeling sets of simple objects describes precisely the correspondence between these two categories. □ Observe that we also get the following theorem as a consequence (see [OW] for similar statements):

Theorem 6.5. *The centralizer algebra $\mathrm{End}_{\mathcal{F}}(V^{\otimes n})$ is generated by the image of $\mathbb{C}\mathcal{B}_n$.*

6.5. **Step 4.** Since any $\mathcal{F} \in \mathfrak{F}$ has the same Groethendieck semiring as any $\mathcal{V} \in \mathfrak{V}$ and the braiding morphism $c_{V,V}$ has 3 distinct eigenvalues, considered as braided tensor categories, the family $\mathfrak{F}$ is a subfamily of $\mathfrak{V}$ by Proposition 5.3. □

6.6. **Extension to Lie type C.** It is known that the Turaev-Wenzl categories of type BC have the same Grothendieck semiring as the categories corresponding to quantum groups of Lie type C at odd roots of unity (see [BB]). Combining this with our result, we get the following rank-level duality type corollary:

Corollary 6.6. *The ribbon categories corresponding to the rank k quantum group of Lie type B and the rank $(\ell-2k-1)/2$ quantum group of Lie type C at a ℓth root of unity have the same tensor product rules.*

Moreover, we can compute the eigenvalues of the braiding isomorphism $c_{V,V}$ for V the highest weight quantum group module of type C_r corresponding to the weight $(1,0,\ldots,0)$. Here it is even easier than for type B as we can use [LR] Corollary 2.22(3). The eigenvalues are:

$$\{q, -q^{-1}, -q^{-2r-1}\}.$$

Using Corollary 6.6 we set $r = (\ell - 2k - 1)/2$ which gives us eigenvalues

$$\{q, -q^{-1}, -q^{\ell}q^{-2k}\}$$

which can be made to match those of $\mathfrak{V}$ by changing an overall sign and/or transposing all Young diagrams as in [TuW2]. Thus we can apply the theorem of Tuba and Wenzl to see that the Lie type C at odd roots of unity categories can be included in this family of ribbon categories.

7. Failure of Unitarity

We will show that no member of the family of braided tensor categories $\mathfrak{V}$ can have the structure of a Unitary ribbon category. We showed in Lemma 4.9 that there is a unique positive character for the Grothendieck semiring $Gr(\mathcal{F})$. By the above equivalence, we also know that the same true for $Gr(\mathcal{V})$ for any $\mathcal{V} \in \mathfrak{V}$. Lemma 5.1 shows that $\dim_{\mathcal{V}}(X)$ is uniquely determined up to a sign by the eigenvalues of the braiding morphism $c_{X,X}$ and so we have that:

$$\pm \dim_{\mathcal{V}}(X) = \frac{-[2k]}{[1]} + 1 \tag{24}$$

So if we can show that $\pm \dim_{\mathcal{V}}(X)$ is never equal to the unique positive character of Lemma 4.9 above for any choice of q^2 a primitive ℓth root of unity then we will have shown that this abstract category does not support both positivity and a braiding. For any $\lambda \in \Gamma(k,\ell)$ we denote by $\mathrm{Dim}(\lambda)$ the unique positive character of $Gr(\mathcal{V})$. Furthermore, we set

$$f_\lambda(z) = \dim_{\mathcal{V}}(X_\lambda) \mid_{(e^{z\pi i/\ell})}$$

for $1 \leq z \leq \ell - 1$ and $\gcd(\ell, z) = 1$ so that $f_\lambda(z)$ takes on all possible values of $\dim_{\mathcal{V}}(X_\lambda)$ as $\mathcal{V}$ ranges over the family $\mathfrak{V}$. We may now formulate:

Theorem 7.1. *If* $2(2k+1) < \ell$ *then* $f_\lambda(z) \neq \mathrm{Dim}(\lambda)$ *for any* z *with* $1 \leq z \leq \ell - 1$ *and* $\gcd(\ell, z) = 1$.

Since both $f_\lambda(z)$ and $\mathrm{Dim}(\lambda)$ are both characters of $Gr(\mathcal{V})$ (i.e. they are normalized so that their values at the trivial object are 1), this theorem will be a consequence of the following:

Lemma 7.2. *Let* $h(z) = f_\Box(z)$. *Then if* $2(2k+1) \leq \ell$ *and* $1 \leq z \leq \ell - 1$ *with* $\gcd(\ell, z) = 1$ *then* $|h(z)| < \mathrm{Dim}(\Box)$.

Proof. We start by showing that $h(z) < \mathrm{Dim}_q(\Box)$. We have that $h(z) = \frac{-\sin(2kz\pi/\ell)}{\sin(z\pi/\ell))} + 1$ and $\mathrm{Dim}_q(\Box) = \frac{\sin((2k+1)\pi/\ell)}{\sin(\pi/\ell)}$. First one notes that $\mathrm{Dim}_q(\Box) > 1$ and so $h(z) < \mathrm{Dim}_q(\Box)$ if $z \leq \ell/2k$. So the lemma is true for $z \in I_1 = [1, \ell/2k]$.

Next we make a change of variables $z \to \ell - z'$ in order to eliminate large z. We define

$$g(z') = h(\ell - z') = \frac{\sin(2kz'\pi/\ell)}{\sin(z'\pi/\ell)} + 1$$

with $1 \leq z' \leq \ell - 1$. Using the sum expansion of $\frac{q^{2k} - q^{-2k}}{q - q^{-1}}$ we can write

$$g(z') = 1 + 2 \sum_{1 \leq j \leq k} \cos((2j-1)z'\pi/\ell).$$

By taking a derivative of $g(z')$ we find that it is a decreasing function of z' on the interval $I_2' = [2, \frac{\ell}{2k-1}]$, which is nonempty if $2(2k-1) \leq \ell$. Thus if $g(2) < \mathrm{Dim}(\Box)$ then $g(z') < \mathrm{Dim}(\Box)$ on all of I_2'. Expanding $\mathrm{Dim}(\Box)$ we compute:

$$\mathrm{Dim}(\Box) - g(2) = 2 \sum_{1 \leq j \leq k} [\cos(2j\pi/\ell) - \cos(2(2j-1)\pi/\ell)].$$

Using the trigonometric formulas found in the back of any calculus book we may express each term $\cos(2j\pi/\ell) - \cos(2(2j-1)\pi/\ell)$ as $2\sin((3j-1)\pi/\ell)\sin((j-1)\pi/\ell)$. Provided $3j - 1 \leq 3k - 1 \leq \ell$, each of these terms is positive. But we already have the stronger restriction $2(2k+1) \leq \ell$, thus we have $g(z') < \mathrm{Dim}(\square)$ on I_2' that is, $h(z) < \mathrm{Dim}(\square)$ on $I_2 = [\ell - \frac{\ell}{2k-1}, \ell - 2]$. We check the case $z' = 1$ separately:

$$\mathrm{Dim}(\square) - g(1) = \sum_{1 \leq j \leq k} [\cos(2j\pi/\ell) - \cos((2j-1)\pi/\ell]$$

and each term can be factored as:

$$-2\sin(\pi/2\ell)\sin((4j-1)\pi/2\ell)$$

which is always strictly negative since $4j - 1 < 2\ell$ for all $j \leq k$.

The only remaining z to eliminate are those between I_1 and I_2. To this end we use the following estimates which come from approximating $\sin(x)$ from below by $1 - |2x/\pi - 1|$ on the interval $0 \leq x \leq \pi$:

$$h(z) < \frac{1}{\sin(z\pi/\ell)} + 1 < 2(2k+1)/\pi \leq \mathrm{Dim}(\square)$$

which are valid for $z \in I_3 = [\frac{\ell\pi}{4(2k+1)-2\pi}, \ell - \frac{\ell\pi}{4(2k+1)-2\pi}]$ provided $2(2k+1) < \ell$. It is now easy to see that $[1, \ell-2] \cup \{\ell - 1\} \subset I_1 \cup I_2 \cup I_3$ thus proving that $h(z) < \mathrm{Dim}(\square)$ for any z, ℓ, k as in the statement.

With a few modifications to this proof we can show that $-h(z) < \mathrm{Dim}_q(\square)$ as follows. On I_3 our estimates are still valid. We observe that $-h(z)$ is decreasing on $[1, \frac{\ell}{2k-1}]$ so one need only check that $\mathrm{Dim}(\square) > -h(1)$, which is straightforward. By changing variables as we did above we can also eliminate $z \in [\ell - \frac{\ell}{2k}, \ell - 2]$ using the observation that $\mathrm{Dim}(\square) > 1$ again. One must again check the case $z = \ell - 1$ separately but the same basic argument works as above except we must use the stronger condition $4k - 1 \leq \ell$ since the factors involved are cosines. □

So we have shown that for no q^2 a primitive ℓth root of unity does the categorical dimension of any $\mathcal{V} \in \mathfrak{V}$ achieve the value of the unique positive character of $Gr(\mathcal{V})$ (or $Gr(\mathcal{F})$). Observe that In order to apply Theorem 2.11 and complete the proof, we observe that there is a simple object $X_\tau \in \mathcal{V}$ with $|\tau|$ even and $\dim_{\mathcal{V}}(X_\tau) < 0$. For if all simple X_τ with $|\tau|$ even had positive dimension by multiplying by $(-1)^{|\lambda|}$ we would get a $\dim_{\mathcal{V}}$ function that was positive on all simple objects but with the same $\dim_{\mathcal{V}}(X_\square)$ up to a sign, which is impossible by the above lemma. Since every X_τ with $|\tau|$ even appears in an even power of X, we can apply Theorem 2.11 and conclude that:

Corollary 7.3. *No braided tensor category $\mathcal{O}$ with $Gr(\mathcal{O}) \cong Gr(\mathcal{V})$ (or $Gr(\mathcal{F})$) is unitarizable.*

Remark 7.4. It should be noted that it was previously thought that the Turaev-Wenzl categories in the BC-case *are* unitary for the choices $q = -e^{\pm\pi i/\ell}$ (see [TW2], Theorem 11.2). The critical theorem used to prove the positivity of the form is in [W1], Theorem 6.4. However, the discovery of a slight miscalculation in the case ℓ odd reveals that the argument fails in the present case.

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