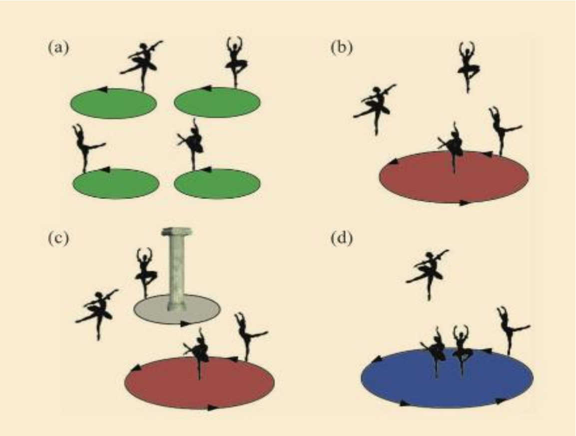
Modeling and Classification of Quantum Hall States

A story of electrons under extreme conditions or of translation invariant symmetric polynomials



Zhenghan Wang
Microsoft Station Q & UC Santa Barbara
Texas, March 23, 2015

Classical Hall Effect---Maxwell's Mistake

SOMETIME during the last University year, while I was reading Maxwell's Electricity and Magnetism in connection with Professor Rowland's lectures, my attention was particularly attracted by the following passage in Vol. II, p. 144:

"It must be carefully remembered, that the mechanical force which urges a conductor carrying a current across the lines of magnetic force, acts, not on the electric current, but on the conductor which carries it. If the conductor be a rotating disk or a fluid it will move in obedience to this force, and this motion may or may not be accompanied with a change of position of the electric current which it carries. But if the current itself be free to choose any path through a fixed solid conductor or a network of wires, then, when a constant magnetic force is made to act on the system, the path of the current through the conductors is not permanently altered, but after certain transient phenomena, called induction currents, have subsided, the distribution of the current will be found to be the same as if no magnetic force were in action. The only force which acts on electric currents is electromotive force, which must be distinguished from the mechanical force which is the subject of this chapter."

The results obtained are as follows:

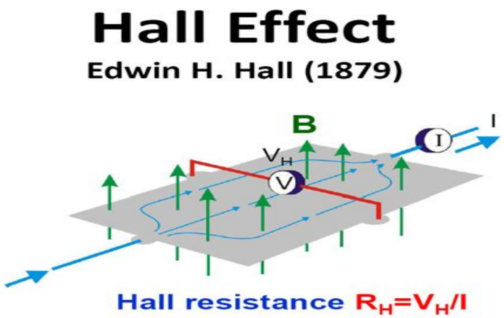
Current through Gold Leaf Strip. C.	Strength of Magnetic Field. M.	Current through Thomson Galvanometer. c.	$\frac{C \times M}{c}$
.0616	11420 H	.0000000232	303000000000.
.0249	11240 "	085	329..........
.0389	11060 "	135	319..........
.0598	7670 "	147	312..........
.0595	5700 "	104	326..........

H is the horizontal intensity of the earth's magnetism = .19 approximately.

When electrons discovered?

On a New Action of the Magnet on Electric Currents
Author(s): E. H. Hall
Source: *American Journal of Mathematics*, Vol. 2, No. 3 (Sep., 1879), pp. 287-292

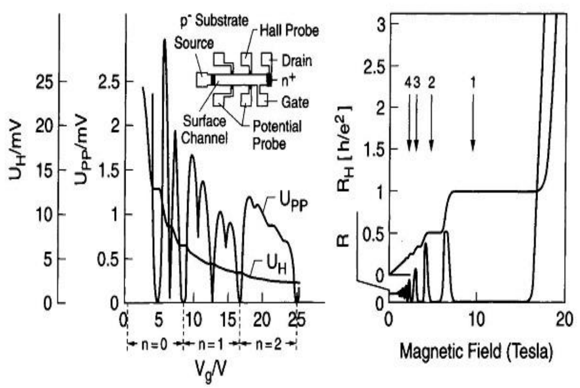
Birth of Integer Quantum Hall Effect



New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance,

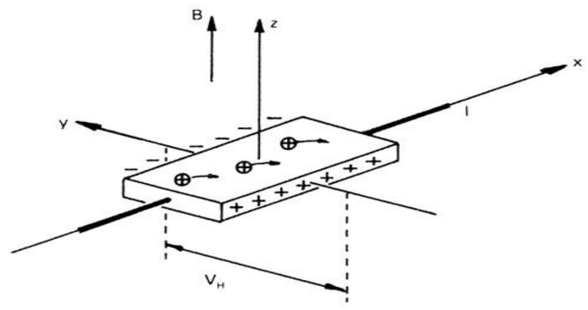
K. v. Klitzing, G. Dorda and M. Pepper

Phys. Rev. Lett. 45, 494 (1980).



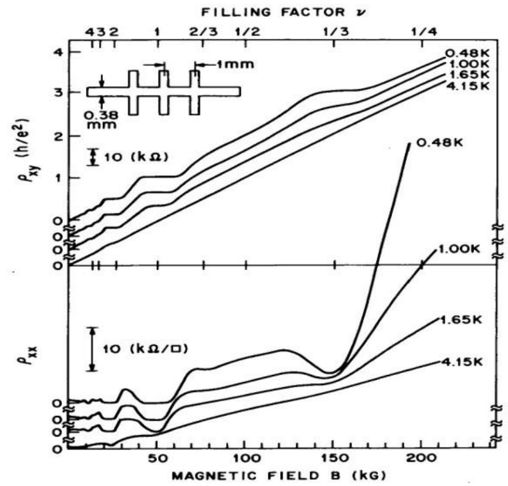
These experimental data, available to the public 3 years before the discovery of the quantum Hall effect, contain already all information of this new quantum effect so that everyone had the chance to make a discovery that led to the Nobel Prize in Physics 1985. The unexpected finding in the night of 4./5.2.1980 was the fact, that the plateau values in the Hall resistance x-y are not influenced by the amount of localized electrons and can be expressed with high precision by the equation $R_H = \frac{h}{\nu e^2}$

Fractional Quantum Hall Effect (FQHE)



D. Tsui enclosed the distance between B=0 and the position of the last IQHE between two fingers of one hand and measured the position of the new feature in this unit. He determined it to be three and exclaimed, “quarks!” H. Stormer

The FQHE is fascinating for a long list of reasons, but it is important, in my view, primarily for one: It established experimentally that both particles carrying an exact fraction of the electron charge e and powerful gauge forces between these particles, two central postulates of the standard model of elementary particles, can arise spontaneously as emergent phenomena. R. Laughlin

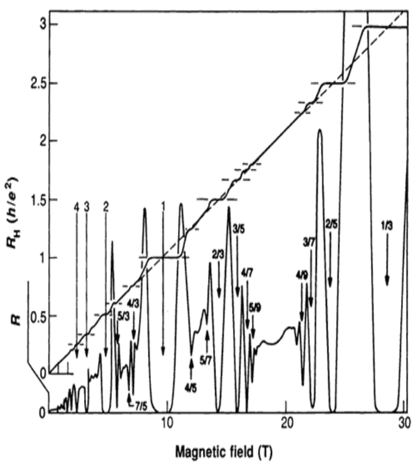


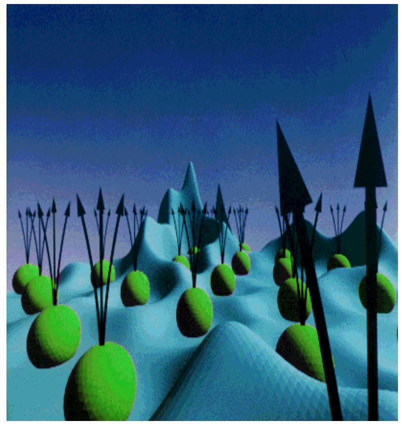
In 1998, Laughlin, Stormer, and Tsui are awarded the Nobel Prize

“ for their discovery of a new form of quantum fluid with fractionally charged excitations.”

D. C. Tsui, H. L. Stormer, and A. C. Gossard
Phys. Rev. Lett. 48, 1559 (1982)

How Many Fractions Have Been Observed? ~100





$\nu = \frac{N_e}{N_\phi}$ filling factor or fraction

N_e = # of electrons

N_ϕ =# of flux quanta

How to model the quantum state(s) at a filling fraction?

What are the electrons doing at a plateau?

1/3	1/5	1/7	1/9	2/11	2/13	2/15	2/17	3/19	5/21	6/23	6/25
2/3	2/5	2/7	2/9	3/11	3/13	4/15	3/17	4/19	10/21		
4/3	3/5	3/7	4/9	4/11	4/13	7/15	4/17	5/19			
5/3	4/5	4/7	5/9	5/11	5/13	8/15	5/17	9/19			
7/3	6/5	5/7	7/9	6/11	6/13	11/15	6/17	10/19			
8/3	7/5	9/7	11/9	7/11	7/13	22/15	8/17				
	8/5	10/7	13/9	8/11	10/13	23/15	9/17				
	11/5	12/7	25/9	16/11	20/13						
	12/5	16/7		17/11							
		19/7									

m/5, m=14,16, 19

5/2

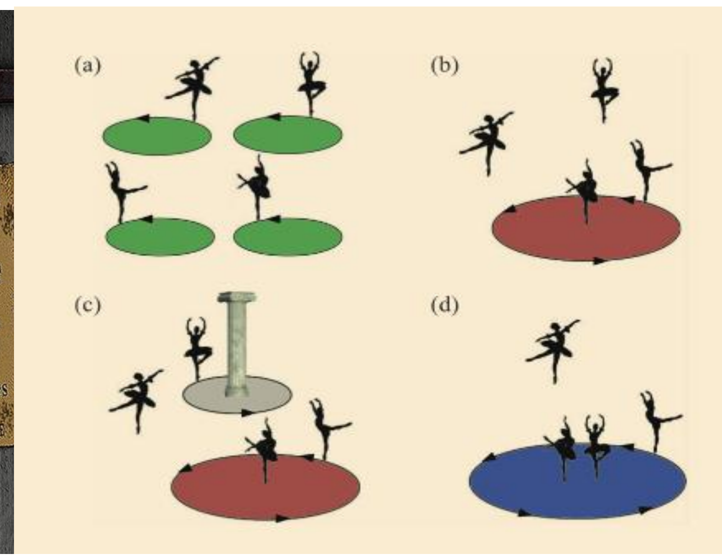
7/2

19/8

Pan et al (2008)

Three Answers

1. Art: electrons find “partners” and dance
2. Physics: patterns of long range entanglement
3. Math: (2+1)-TQFT or modular tensor category in nature



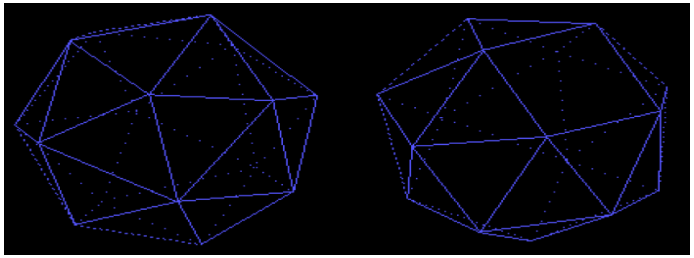
State and Energy

- At each moment, a physical system is in some state-a point in R^n or a vector $(\neq 0)$ in some Hilbert space (a wave function).
- Each state has an energy.
- States of the lowest energy win: stable

How to find the lowest energy states (ground states) and understand their properties (excitations=responses)?

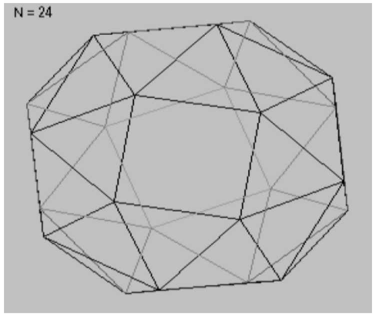
Classical Electrons on S^2

Thomson's Problem:



Stable configurations of N-electrons on S^2 minimizing total Coulomb potential energy

$E_{ij} = \sum_{i \neq j} \frac{1}{d_{ij}}$, d_{ij}=distance between i, j



What happens if $N \to \infty$?

Mathematical Quantum Systems

- A triple Q=(L, H, c), where L is a Hilbert space with a preferred basis c, and H an Hermitian matrix.

  Physically, H is the Hamiltonian. A non-zero vector in L is a quantum state = wave function.
- Given a quantum system, find the ground state manifold: the eigenspace of H with the smallest eigenvalue:

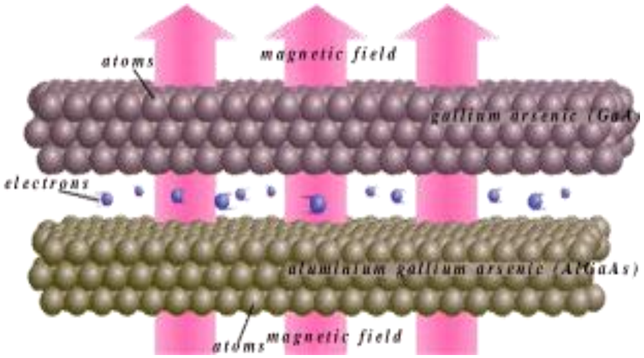
  $$\mathbf{L=\oplus V_i},$$

  where V_i is the eigenspace of H with eigenvalue=energy $\lambda_i, \mathrm{i} = 0,1, \ldots$, in an increasing order.

  V_0 is the ground state manifold with energy=λ_0 and others are excited states.
- A linear algebra problem that needs a quantum computer.

Quantum Hall Systems

N electrons in a plane bound to the interface between two semiconductors immersed in a perpendicular magnetic field



Phases are equivalence classes of ground state wave functions that have similar properties or no phase transitions as N→∞ (N ~ $10^{11}\ cm^{-2}$)

Fundamental Hamiltonian:

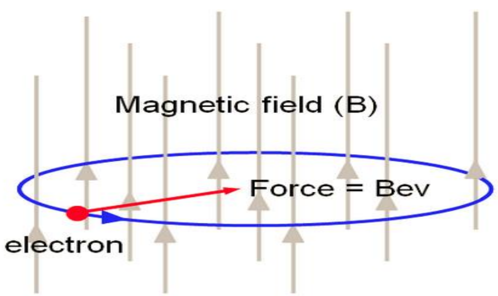
H =$\Sigma_1{}^N \{\frac{1}{2m}\ [\nabla_j - q\ A(z_j)]^2 + V_{bg}(z_j)\} + \Sigma_{j<k} V(z_j - z_k)$

Model Hamiltonian:

H=$\Sigma_1{}^N\{\frac{1}{2m}\ [\nabla_j - q\ A(z_j)]^2\} +$ **?**, e.g. $\Sigma_{j<k}\ \delta(z_j - z_k)$, z_j position of j-th electron

Electrons in Plane

- Technology made 2D possible
- Coulomb potential is translation invariant
- Pauli exclusion principle: spin degeneracy
- Spin deg. resolved by magnetic field



Lorentz force:
$F = q\, v \times B$

Quantum phases of matter at T~0.

Many Electrons in a Magnetic Field

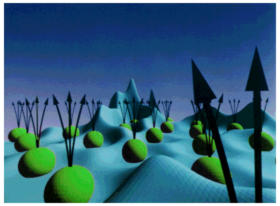
- Landau solution: electron at position z,

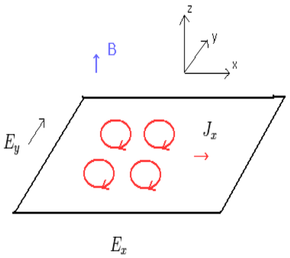
  single electron wave function $\psi_m = z^m e^{-\frac{1}{4}|z|^2}$,

  many electrons $p(z)e^{-\frac{1}{4}|z|^2}$,

  $p(z)$ = polynomial---describe how electrons organize themselves under extreme conditions

- $\nu = 1, p(z_1, z_2, \ldots, z_N) = \prod_{i<j}(\mathrm{z_i} - \mathrm{z_j})$.
- p(z) for $\nu = \frac{1}{3}$?





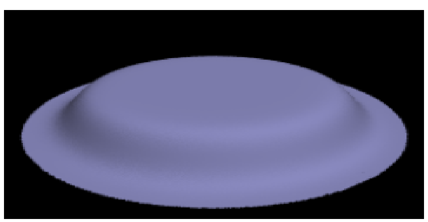
Laughlin State for ν=1/3

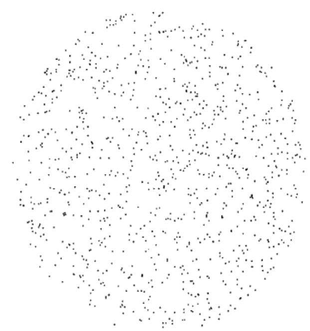
Laughlin 1983, Nobel 1998

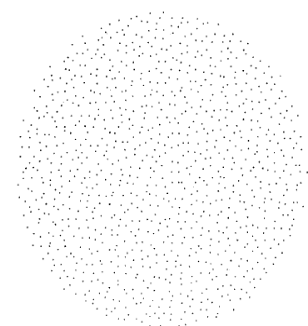
N electrons at z_i in ground state

Gaussian

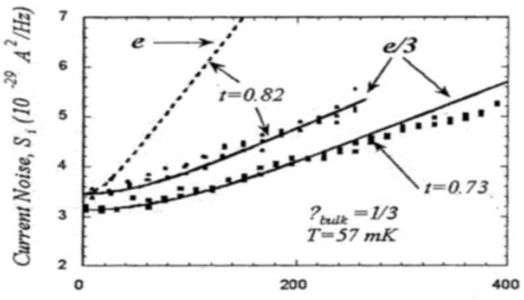
$$\Psi_{1/3} = \prod_{i<j} (z_i - z_j)^3 \, e^{-\Sigma_i |z_i|^2/4}$$







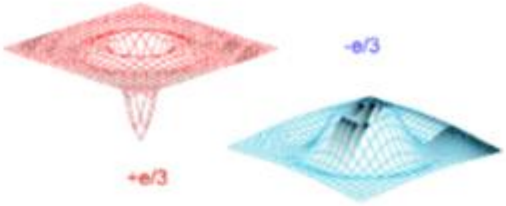
Laughlin Right?



Physical Predictions:

1. Elementary excitations have charge e/3 (Laughlin 83, Nobel 98)
2. Elementary excitations are abelian anyons (Arovas-Schrieffer-Wilczek 84)

Experiments:



Laughlin wave function is a good model

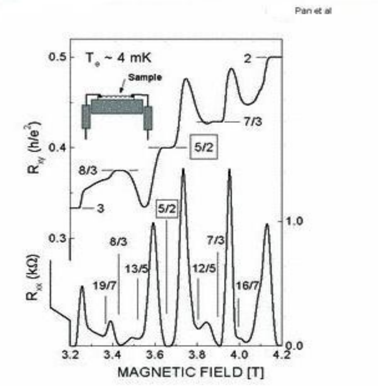
Enigma of ν=5/2 FQHE

R. Willett et al discovered ν=5/2 in1987

- Moore-Read State, Wen 1991
- Greiter-Wilczek-Wen 1991
- Nayak-Wilczek 1996
- Morf 1998
- …

MR (maybe some variation) is a good trial state for 5/2

- Bonderson, Gurarie, Nayak 2011,



Willett et al, PRL 59 1987

A landmark (physical) proof for the MR state

"Now we eagerly await the next great step: experimental confirmation." ---Wilczek

Experimental confirmation of 5/2:

charge e/4 √ , but non-abelian anyons ???

Pfaffian State

G. Moore, N. Read 1991

Pfaffian state (MR w/ ≈ charge sector)

$$\Psi_{Pf}=\mathrm{Pf}(1/(z_i-z_j)) \prod_{i<j}(z_i-z_j)^2\ e^{-\Sigma_i|z_i|^2/4}$$

Pfaffian of a 2n×2n anti-symmetric matrix M=(a_{ij}) is

ω^n =n! Pf (M) $dx^1\wedge dx^2\wedge\ldots\wedge dx^{2n}$ if $\omega=\Sigma_{i<j}\ a_{ij}\ dx^i\wedge\ dx^j$

Physical Theorem:

Elementary excitations are non-abelian anyons, called Ising anyon σ

…… Read 09

A Mathematical Classification

joint work with X.-G. Wen (MIT and PI)

- How to label UNIQUELY a fractional quantum Hall (FQH) state?

  A collection of model wave functions $\{\Psi_k\}$---

  classification of FQH states.

- How to calculate topological properties of FQH states from wave functions?

  E.g. Statistics of anyons=unitary representations

  of the braid groups.

Wave Function of Bosonic FQH State

- Chirality:

  $p(z_1,\ldots,z_N)$ is a polynomial (Ignore Gaussian)
- Statistics:

  symmetric=anti-symmetric divided by $\Pi_{i<j}(z_i-z_j)$
- Translation invariant:

  $p(z_1+c,\ldots,z_N+c) = p(z_1,\ldots,z_N)$ for any $c \in \mathbb{C}$
- Filling fraction:

  $\nu=\lim \frac{N}{N_\phi}$, where N_ϕ is max degree of any z_i

Polynomial of Infinite Variables

- A sequence of translation invariant symmetric polynomials $\{\mathrm{P_k} = p(z_1, \ldots, z_{N_k})\}$ is called a ν-polynomial of infinite variables if there is a positive $\nu \in \mathbb{Q}$ such that $lim_{k\to\infty} \frac{N_k}{d_k} = \nu$, where $d_k =$ maximum degree of z_1 in P_k
- $p(z_1, \ldots, z_{N_k})$ is a model wave function of N_k electrons in a magnetic field.
- When a μ-polynomial of infinite variables represents a FQH state?

Examples

Laughlin: ν=1/q, $N_k = k$, q=even

$$P_{k,1/q} = \prod_{i<j}(z_i - z_j)^q$$

Pfaffian: $\nu = 1/q$, $N_k = 2k$, q = odd

$$P_{k,Pf} = \mathrm{Pf}(1/(z_i - z_j)) \prod_{i<j}(z_i - z_j)^q$$

General Constructions

Given $f(z_1, \ldots, z_N)$ polynomial

- Symmetrization:

  $S(f)(z_1, \ldots., z_N) = \sum_{\sigma \in S_N} f(z_{\sigma(1)}, \ldots, z_{\sigma(N)})$

  is symmetric. Note $S(z_1 - z_2) = 0$.

- Center-of-mass substitution:

  $\mathrm{T}(f)(z_1, \ldots., z_N) = f(z_1^{(c)}, \ldots, z_N^{(c)})$,

  $z_i^{(c)} = \mathrm{z_i} - \frac{\mathrm{z_1} + \cdots + \mathrm{z_N}}{\mathrm{N}}$. Note $\mathrm{T}(z_1 + z_2) = 0$.

Pauli Exclusion Principle

- $|\Psi_k|^2$ is probability
- The probability for two electrons at the same position is zero, so $p(z_1, \ldots, z_{N_k})$ = 0 whenever $z_i = z_j$ for some $i \neq j$. But this is encoded in the Vandermonde factor $\prod(z_i - z_j)$.
- How about more than $a > 2$ electrons in the same position?

Poly. $\{p(z_1, \ldots, z_{N_k})\}$ "vanish" at certain powers $\{S_a\}$ when a particles are brought together

Pattern of Zeros=Quantified Generalized Pauli Exclusion Principle

Given poly. $\{p(z_1, \ldots, z_{N_k})\}$:

$p\left(z_1, \ldots, z_{N_k}\right) = \sum_I c_I z^I$, $\mathrm{I} = (\mathrm{i}_1, \ldots, \mathrm{i}_\mathrm{n})$, $z^I = z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}$

$S_{a,k}$=min$\{\sum_{j=1}^{a} i_j\}$---minimal total degrees of a variables.

If $S_{a,k} = S_a$ for all k such that $\mathrm{N_k} \geq a$, then the sequence $\{S_a\}$ of integers is called the **pattern of zeros** (POZ) of the polynomial of infinite variables.

Morally, $\{S_a\} \approx$ model wave function and encode many topological properties of the FQH state.

CFT Examples

Laughlin: S_a=qa(a-1)/2, ν=1/q, $N_k = k$

$$P_{1/q} = \prod_{i<j}(z_i - z_j)^q$$

Pfaffian: S_a=a(a-1)/2-[a/2], ν=1, $N_k = 2k$

$$P_{1/2} = \mathrm{Pf}(1/(z_i - z_j)) \prod_{i<j}(z_i - z_j)$$

In a CFT, if V_e is chosen as the electron operator and a conformal block as a W.F.

If $V_a=(V_e)^a$ has scaling dimension h_a, then

$$S_a = h_a - a\, h_1$$

Quantum Hall State

- A **quantum Hall state** at filling fraction ν is a ν-polynomial of infinite variables **which satisfies the UFC and nCF conditions and whose POZ has even $\boldsymbol{\Delta_3}$.**
- Classification:

1) find all possible POZs of FQH states,
2) realize them with polynomials,
3) when POZs are FQH states?

Fuse a-electrons

Given a-electrons at $\{z_i, i = 1, \dots, a\}$

set $z_i = z_1^a + \lambda\xi_i$,

where $z_1^a = (\sum_i z_i)/a$, and $\sum|\xi_i|^2=1$.

Imagine z_i as vertices of a simplex, then z_1^a is the barycenter of the simplex. As $\lambda \to 0$, $z_i \to z_1^a$ keeping the same shape.

Sphere S^{2a-3} of $\{\xi_i\}$ parameterizes the shape of the a-electrons.

Unique Fusion Condition

Take a-variables z_i fusing them to $z_1^{(a)}$

The resulting polynomials (coefficients of λ^k)

$\mathrm{p_k}(\mathrm{z}_1^{\mathrm{a}}, \xi_1, \ldots, \xi_a;\ z_{a+1}, \ldots, z_n)$

depend on the shape $\{\xi_i\} \in S^{2a-3}$ of $\{z_i\}$.

If the resulting polynomials of $z_1^{(a)}, z_{a+1}, \ldots, z_n$

for each k of $\mathrm{p_k}(\xi_i)$ span $\leq$1-dim vector space, the poly. satisfies UFC.

Derived Polynomials

Given $p(z_1, \ldots z_N)$, if all variables are fused to new variables $z_i^{(a)}$ and UFC is satisfied, then the resulting new polynomial $p(z_i^a)$ is well-defined, and called the derived polynomial.

Derived polynomials for Laughlin states:

$$\prod_{a<b}\prod_{i,j}(z_i^a - z_j^b)^{qab}\prod_a\prod_{i<j}(z_i^a - z_j^a)^{qa^2}$$

n-cluster Form

If there exists an n>0 such that for any $k, n|N_k$, then the derived polynomial of n-clusters is

$$\prod_{a<b}(z_a^n - z_b^n)^Q$$

The poly. has the n-cluster form (nCF)

nCF reduces pattern of zeros to a finite problem:

$S_{a+kn}=S_a+kS_n+kma+k(k-1)mn/2$, where ν=n/m.

Pattern of Zeros Classification

Theorem (Wen-W.)

If a ν-polynomial of infinite variable $\{p(z_i)\}$ satisfy UFC and nCF for n, set $m=S_{n+1}-S_n$. Then

1) mn even, and $\nu=n/m$

2) $S_{a+b}-S_a-S_b \geq 0$

3) $S_{a+b+c}-S_{a+b}-S_{b+c}-S_{c+a}+S_a+S_b+S_c \geq 0$

4) S_{2a} even

5) $2S_n=0 \mod n$

6) $S_{a+kn}=S_a+kS_n+kma+k(k-1)mn/2$

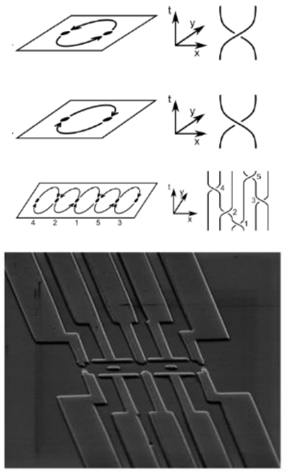
POZ is not complete data for FQH state.

Puzzle: Need $\Delta_3 = S_{a+b+c}-S_{a+b}-S_{b+c}-S_{c+a}+S_a+S_b+S_c$ to be EVEN!

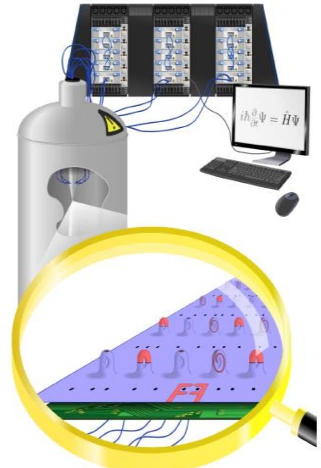
What Are Polys of Infinite Variables?

- Represent topological phases of matter.
- Universal properties of topological phases of matter are encoded by TQFTs/modular tensor categories/CFTs, so polynomials of infinite variables are TQFTs/MTCs/CFTs.

How does this connection manifest, e.g. how to derive MTCs/CFTs from POZs?



(2+1)-TQFT



Modular Tensor Category (MTC)

Topological Phase of Matter

Topological Quantum Computation

Topological phases of matter are TQFTs in Nature and hardware for hypothetical topological quantum computers.