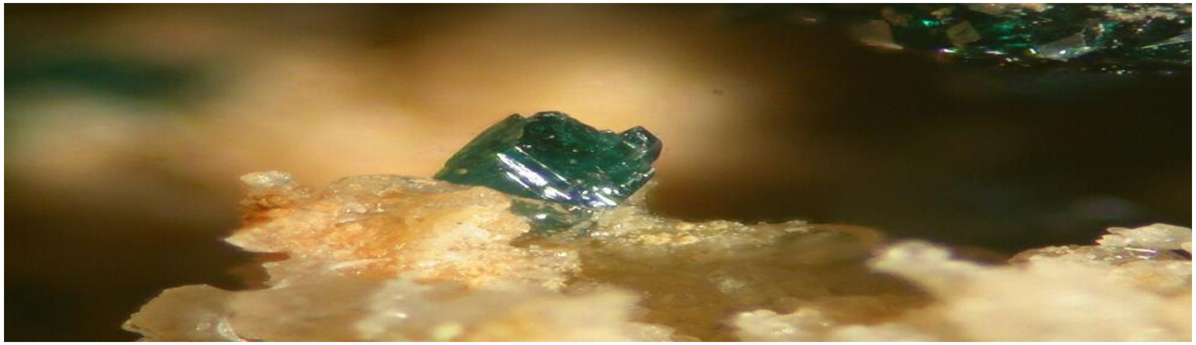
Modeling and Classification of Topological Phases of Matter



Zhenghan Wang
Microsoft Station Q & UC Santa Barbara
Texas, March 24, 2015

Microsoft Station Q

Search for non-abelian anyons in topological phases of matter, and build a topological quantum computer

Theory: Station Q,…

Experiment: Delft, Copenhagen,…

Computer Science: QuArc

Mike Freedman (director, math-Fields medalist), Chetan Nayak (physics), Kevin Walker (math), Matt Hastings (physics), Parsa Bonderson (physics), Roman Lutchyn (physics), Bela Bauer (physics)

+ collaborators
+ postdocs
+graduate students
+visitors

http://stationq.ucsb.edu/

As a mathematician working also in physics…

COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. XIII, 001–14 (1960)

The Unreasonable Effectiveness of Mathematics in the Natural Sciences

Richard Courant Lecture in Mathematical Sciences delivered at New York University, May 11, 1959

EUGENE P. WIGNER
Princeton University

For his contributions to the theory of the atomic nucleus
and the elementary particles, particularly through the discovery
and application of **fundamental symmetry principles**

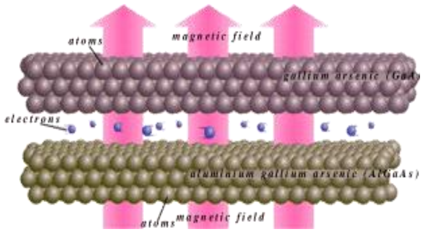
Convergence of Physics and Mathematics

Physics	Mathematics
Newtonian Mechanics	Calculus (arranged marriage)
General Relativity and Gauge theory	Differential Geometry
Quantum Mechanics	Linear Algebra and Functional Analysis
Many-body Entanglement Physics (second revolution in quantum mechanics?)	???

Universal Properties of 2D Topological Phases

Topological Quantum Field Theory (TQFT) and Modular Tensor Category (MTC)

2D Topological Phases in Nature



- Quantum Hall States

  1980 Integral Quantum Hall Effect ---

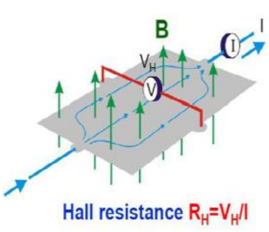
  von Klitzing (1985 Nobel)

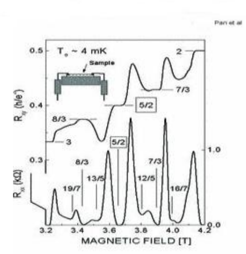
  1982 Fractional QHE---Stormer, Tsui, Gossard at $\nu = \frac{1}{3}$

  (1998 Nobel for Stormer, Tsui, and Laughlin)

  1987 Non-abelian FQHE**???**---R. Willett et al at $\nu = \frac{5}{2}$

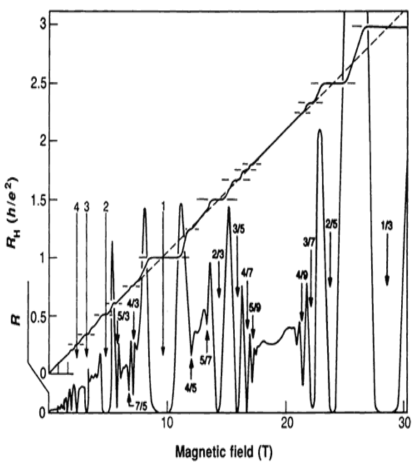
- Topological superconductors
- Topological insulators
- …

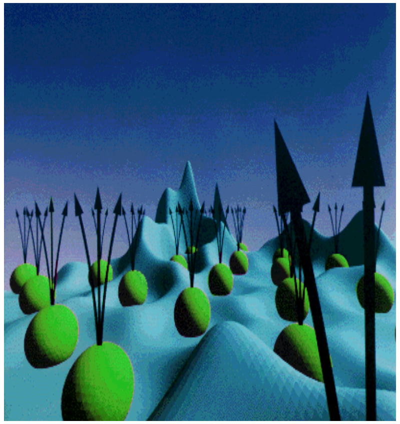




$R_H = \nu^{-1}\frac{h}{e^2}$

~100 Fractional Quantum Hall States





$\nu = \frac{N_e}{N_\phi}$ filling factor or fraction

N_e = # of electrons

N_ϕ =# of flux quanta

Topological phases at a filling fraction are modeled by TQFTs/UMTCs/CFTs/…
(or pattern of zeros)

1/3 1/5 1/7 1/9 2/11 2/13 2/15 2/17 3/19 5/21 6/23 6/25
2/3 2/5 2/7 2/9 3/11 3/13 4/15 3/17 4/19 10/21
4/3 3/5 3/7 4/9 4/11 4/13 7/15 4/17 5/19
5/3 4/5 4/7 5/9 5/11 5/13 8/15 5/17 9/19
7/3 6/5 5/7 7/9 6/11 6/13 11/15 6/17 10/19
8/3 7/5 9/7 11/9 7/11 7/13 22/15 8/17
8/5 10/7 13/9 8/11 10/13 23/15 9/17
11/5 12/7 25/9 16/11 20/13
12/5 16/7 17/11
19/7
m/5, m=14,16, 19

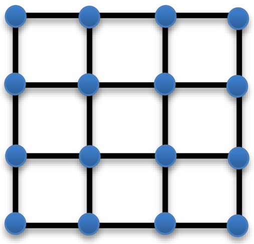
5/2
7/2
19/8

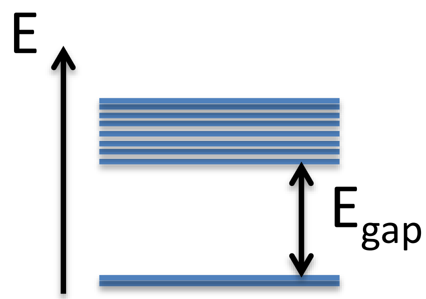
Pan et al (2008)

Topological Phases of Quantum Matter

Local Hilbert Space $L = \bigotimes_i L_i$

Local, Gapped Hamiltonian $H: L \to L$





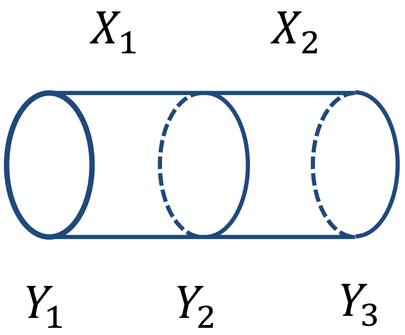
Two gapped Hamiltonians H_1, H_2 realize the same phase of matter if there exists a continuous path connecting them without closing the gap/a phase transition.

A topological phase, to first approximation, is a class of gapped Hamiltonians that realize the same phase. Topological order in a 2D topological phase is encoded by a TQFT or anyon model=unitary modular tensor category (MTC) or CFT.

Atiyah-Segal Type (2+1)-TQFT: Codim=1

A symmetric monoidal "functor" (V, Z):
category of 2-3-mfds → Vec
2-mfd Y → vector space $V(Y)$
3-bord X from Y_1 to Y_2 → Z(X): $V(Y_1) \to V(Y_2)$

- $V(\emptyset) = \mathbb{C}$
- $V(Y_1 \sqcup Y_2) \cong V(Y_1) \otimes V(Y_2)$
- $V(-Y) \cong V^*(Y)$
- $Z(Y \times I) = \mathrm{Id}_{V(Y)}$
- $Z(X_1 \cup X_2) = \kappa^m \cdot Z(X_1) \cdot Z(X_2)$ (anomaly)



TQFTs and Higher Category Theories

Basic Principle:
Physics is local, so realistic TQFTs are determined by local data.

(n+1)-Topological Quantum Field Theories ← – – → (n+1)-Categories

(2+1)-TQFTs ⟷ Modular Tensor Categories

Quantum Finite Group Algebras

1. Not fully extended. Not covered by Lurie's cobordism hypothesis.
2. Frontiers are in (3+1)D both mathematically and physically:
(2+1)-TQFTs are unemployed---no major topological problems to solve,
(3+1)-TQFTs that can detect smooth structures are highly desired.

Generalization of Two Theorems

- **Landau's Theorem**:

  Given r, there are only finitely many groups with exactly r irreducible representations.

- **Cauchy Theorem**:

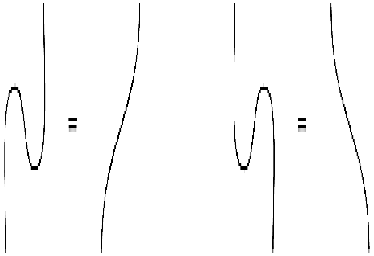
  Given a finite group G, the prime factors of the order of G and the exponent of G are the same set.

A modular (tensor) category is a spherical fusion category with a non-degenerate braiding

A fusion category is a categorification of a based ring $\mathbb{Z}[\boldsymbol{x_0}, \dots, \boldsymbol{x_{r-1}}]$

finite rigid $\mathbb{C}$-linear semisimple monoidal category with simple unit

monoidal: $(\otimes, \mathbf{1})$,
semisimple: $X \cong \bigoplus_i m_i X_i$,
linear: $\mathrm{Hom}(X, Y) \in \mathrm{Vec}_{\mathbb{C}}$,
rigid: $X^* \otimes X \mapsto \mathbf{1} \mapsto X \otimes X^*$



finite rank: $\mathrm{Irr}(\mathcal{C}) = \{\mathbf{1} = X_0, \dots, X_{r-1}\}$

$\boldsymbol{X}$ **simple if** $\mathbf{Hom}(\boldsymbol{X}, \boldsymbol{X}) = \mathbb{C}$

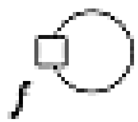
Rank of $\mathcal{C}$: $\mathrm{r}(\mathcal{C}) = r = \dim V(T^2)$

Spherical Fusion Category

- Rigidity defines a functor $^{**}: \mathcal{C} \to \mathcal{C}$. **A pivotal structure** is a natural isomorphism between the identity functor Id_{C} and **.

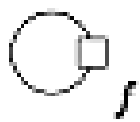
  Define a left trace and a right trace for any morphism $f: x \to x$:

$$\mathrm{Tr}^r(f) = d_{x^*} \circ (\phi_x \otimes \mathrm{id}_{x^*}) \circ (f \otimes \mathrm{id}_{x^*}) \circ b_x$$



$$= \quad : 1 \xrightarrow{b_x} x \otimes x^* \xrightarrow{f \otimes \mathrm{id}_{x^*}} x \otimes x^* \xrightarrow{\phi_x \otimes \mathrm{id}_{x^*}} x^{**} \otimes x^* \xrightarrow{d_{x^*}} 1$$

$$\mathrm{Tr}^l(f) = d_x \circ (d_{x^*} \otimes f) \circ (\mathrm{id} \otimes \phi_x^{-1}) \circ b_{x^*}$$



$$= \quad : 1 \xrightarrow{b_{x^*}} x^* \otimes x^{**} \xrightarrow{\mathrm{id} \otimes \phi_x^{-1}} x^* \otimes x \xrightarrow{\mathrm{id}_{x^*} \otimes f} x^* \otimes x \xrightarrow{d_x} 1$$

- A pivotal structure is **spherical** if the two traces are equal.
- **A fundamental open question:**

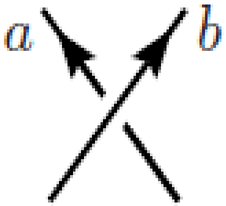
  Is every fusion category pivotal/spherical?

Modular Category

A fusion category is braided if there exist braidings

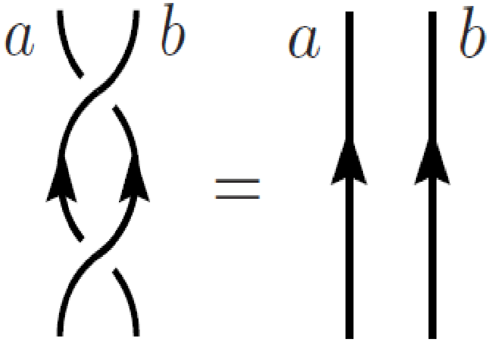
$c_{a,b}: a \otimes b \longrightarrow b \otimes a$

satisfy hexagons.



A simple object a is **transparent** if for any simple b,

$c_{b,a} \cdot c_{a,b} = \mathrm{id}_{a \otimes b}.$



A braiding is **non-degenerate** if the only transparent simple is the tensor unit.

Examples

- Pointed: $\mathcal{C}(A, q)$, A finite abelian group, q non-degenerate quadratic form on A.
- $\mathrm{Rep}(D^\omega G)$, ω a 3-cocycle on G a finite group.
- Quantum groups/Kac-Moody algebras: subquotients of $\mathrm{Rep}(U_q\mathfrak{g})$ at $q = e^{\pi i/l}$ or level k integrable $\hat{\mathfrak{g}}$-modules, e.g.
  - $\mathrm{SU}(N)_k = \mathcal{C}(\mathfrak{sl}_N, N + k)$,
  - $\mathrm{SO}(N)_k$,
  - $\mathrm{Sp}(N)_k$,
  - for $\gcd(N, k) = 1$, $\mathrm{PSU(N)_k} \subset \mathrm{SU}(N)_k$ "even half"
- Drinfeld center: $\mathcal{Z}(\mathcal{D})$ for spherical fusion category $\mathcal{D}$.

Invariants of Modular Tensor Category

MTC $\mathcal{C}$ ⇄ (2+1)-TQFT (V, Z)

- Pairing $\langle Y^2, \mathcal{C}\rangle = V(Y^2; \mathcal{C}) \in \mathrm{Rep}(\mathcal{M}(Y^2))$ for a surface Y^2, $\mathcal{M}(Y^2)$ = mapping class group
- Pairing $Z_{X,L,\mathcal{C}} = \langle (X^3, L_C), \mathcal{C}\rangle \in \mathbb{C}$ for colored framed oriented links L_c in 3-mfd X^3

fix $\mathcal{C}$, $Z_{X,L,\mathcal{C}}$ invariant of (X^3, L_c)

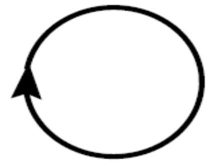
fix (X^3, L_c), $Z_{X,L,\mathcal{C}}$ invariant of $\mathcal{C}$

fix Y^2, $V(Y^2; \mathcal{C})$ invariant of $\mathcal{C}$

Quantum Dimensions and Twists: Unknot

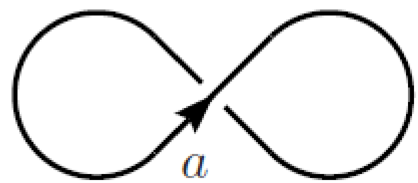
- Label set $L =$ isomorphism classes of simple objects
- Quantum dimension of a simple/label $a \in L$:

$$d_a = d_{\bar{a}} = a$$



- Topological twist/spin of a: finite order by Vafa's thm

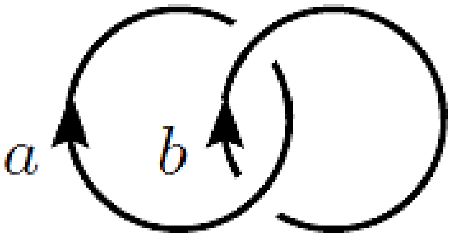
$$\theta_a = \theta_{\bar{a}} = \frac{1}{d_a}$$



- Dimension D^2 of a modular category:

$$\dim(\mathcal{C}) = \boldsymbol{D^2 = \sum_{a \in L} d_a^2}$$

Modular S-Matrix: Hopf Link

- Modular S-matrix: $S_{ab} =$ 

- Modular T-matrix: $T_{ab} = \delta_{ab}\theta_a$-diagonal

- (S, T)-form a projective rep. of $SL(2, \mathbb{Z})$:

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow \mathrm{S}$$

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rightarrow \mathrm{T}$$

Modular Data

1. $S = S^t, S\bar{S}^t = D^2\mathrm{Id}$, modular T diagonal, $\mathrm{ord}(T) = N < \infty$
2. $(ST)^3 = p_+S^2, p_+p_- = D^2, \left(\frac{p_+}{p_-}\right)^N = 1$
3. $N_{ij}^k = \sum_a \frac{S_{ia}S_{ja}\overline{S_{ka}}}{D^2 d_a} \in \mathbb{N}$, Verlinde formulas for fusion rules
4. $\theta_i\theta_j S_{ij} = \sum_k N_{i^*j}^k d_k \theta_k$,where $N_{ii^*}^0$ uniquely defines i^*.
5. $\nu_n(k) := \frac{1}{D^2}\sum_{i,j} N_{ij}^k d_i d_j \left(\frac{\theta_i}{\theta_j}\right)^n \in \mathbb{Z}[e^{\frac{2\pi i}{N}}]$ satisfies:
   $\nu_2(k) \in \{0, \pm 1\}$
6. $\mathbb{Q}(S) \subset \mathbb{Q}(T)$, $\mathrm{Aut}_{\mathbb{Q}}\mathbb{Q}(S) \subset \mathfrak{S}_r$, $\mathrm{Aut}_{\mathbb{Q}(S)}\mathbb{Q}(T) \cong (\mathbb{Z}_2)^k$

$$p_\pm := \sum_j d_j^2 \theta_j^{\pm 1} \qquad N_{ij}^k = \dim \mathrm{Hom}(X_i \otimes X_j, X_k)$$

Rank-Finiteness

Theorem (Bruillard-Ng-Rowell-W., 2013):

For a fixed rank, there are only finitely many equivalence classes of modular categories.

Remarks

1. Refinement of Ocneanu rigidity: fix the fusion rule, finite.
2. Rank-finiteness for fusion/spherical fusion categories open.
3. An explicit bound and effective algorithm.

Finite Group Analogue

Theorem (E. Landau 1903)
For any $r \in \mathbb{N}$, *there are finitely many groups* G *with* $|\mathrm{Irr}(G)| = r$.

Proof.
Use class equation:

$$|G| = \sum_{i=1}^{r} |\bar{g}_i|,$$

$\bar{g}_i$ distinct conjugacy classes. Set $x_i = [G:C(g_i)]$ (index of centralizers) to get

$$1 = \sum_{i=1}^{r} \frac{1}{x_i}.$$

$x_i \leq a(r)$ where $a(1) = 2$, $a(2) = 3$, $a(n) = a(n-1)a(n-2) + 1$ is Sylvester's sequence. Therefore $|G| = \max_i x_i$ is bounded. □

Dimension Equation

Fix rank r, $\dim(\mathcal{C}) = D^2 = d_0^2 + d_1^2 + \cdots + d_{r-1}^2$, $d_0 = 1$

Rewrite: $1 - \boldsymbol{D^2} + \boldsymbol{d_1^2} + \cdots + \boldsymbol{d_{r-1}^2} = 0$

Quantum dimensions $\boldsymbol{d_a^2}$ and $\boldsymbol{D^2}$ are special algebraic integers:

$\mathcal{S}$-units

Let $\mathbb{K}$ be a number field and $\mathcal{S} \in \operatorname{Spec} \mathcal{O}_{\mathbb{K}}$ be finite. The $\mathcal{S}$-units:

$$\mathcal{O}_{\mathbb{K},\mathcal{S}}^{\times} = \{x \in \mathbb{K} \mid \langle x \rangle = \prod_{\mathfrak{p} \in \mathcal{S}} \mathfrak{p}^{\alpha_{\mathfrak{p}}}\}$$

where $\alpha_{\mathfrak{p}} \in \mathbb{Z}$.

$$\left(\mathcal{O}_{\mathbb{K},\mathcal{S}}^{\times} = \{x \in \mathbb{K}^{\times} \mid \|x\|_v = 1 \text{ for all } v \notin \mathcal{S}\}\right)$$

Reduction to Evertse's Theorem

Theorem (Evertse 1984)
There are finitely many solutions to $0 = 1 + x_0 + \cdots + x_{r-1}$ *with* $x_i \in \mathcal{O}_{\mathbb{K},\mathcal{S}}^{\times}$ *such that no sub-sum of* $1 + x_0 + \cdots + x_{r-1}$ *vanishes.*

Set $m = \operatorname{lcm}(\operatorname{ord}(T))$ for all rank $= r$ modular T, $\mathbb{K} = \mathbb{Q}(e^{\frac{2\pi i}{m}})$.
$\mathcal{S} = \{s_i \in \operatorname{Spec}(\mathcal{O}_{\mathbb{K}}) \mid s_i | \wp \in \mathcal{M}_r ???\}$

Evertse's Theorem implies:

$$|\{(-\dim(\mathcal{C}), (d_1)^2, \ldots, (d_{r-1})^2)\}| < \infty.$$

Hence $\dim(\mathcal{C})$ is bounded. By Verlinde formulas, only finitely many fusion rules. Rank-finiteness follows from Ocneanu rigidity.

Need to show: $\mathcal{M}_r$ is finite, and $\boldsymbol{d_a^2}$ and $\mathbf{dim}(\boldsymbol{\mathcal{C}})$ are **$\boldsymbol{\mathcal{S}}$-units**

Prime Factorization

- For a rank $= r$ modular category $\mathcal{C}$, N=ord(T), modular T:

$$\mathcal{S}_{\mathcal{C}} = \{\wp \in \operatorname{Spec}\big(\mathbb{Z}(\zeta_N)\big) \mid \wp | \langle \dim(\mathcal{C}) \rangle\}, \text{ and}$$

$$\mathcal{M}_{\mathcal{C}} = \{\wp \in \operatorname{Spec}\big(\mathbb{Z}(\zeta_N)\big) \mid \wp | \langle \operatorname{ord}(T) \rangle\}.$$

- For a fixed rank r,

$$\mathcal{S}_r := \bigcup_{\operatorname{rank}(\mathcal{C})=r} \mathcal{S}_{\mathcal{C}}$$

and

$$\mathcal{M}_r := \bigcup_{\operatorname{rank}(\mathcal{C})=r} \mathcal{M}_{\mathcal{C}}$$

Cauchy Theorem for Modular Categories

Theorem (Bruillard-Ng-Rowell-W.)

$\boldsymbol{\mathcal{M}_{\mathcal{C}} = \mathcal{S}_{\mathcal{C}}}$**, i.e. prime divisors of** $\boldsymbol{\dim(\mathcal{C})}$ **and N =** **ord**$(\boldsymbol{T})$ **in** $\mathbb{Z}(\zeta_N)$ **form the same set.**

- $\mathcal{M}_r = \mathcal{S}_r$ for any rank r, d_a^2 and $\dim(\mathcal{C})$ are **$\mathcal{S}$-units**.
- Finiteness of $\mathcal{M}_r$ follows from Ng-Schauenburg congruence subgroup theorem for modular rep. of $\mathrm{SL}(2, \mathbb{Z})$.

Classification of Unitary Modular Categories

rank $= 2, 3, 4$ with Rowell and Stong, rank $= 5$ with Bruillard, Ng, Rowell

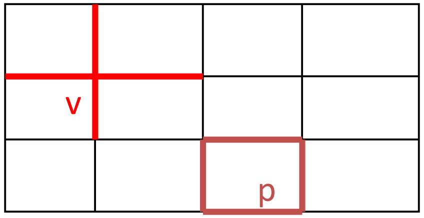
	A 1 Trivial			
	A 2 Semion		NA 2 Fib BU	
	A 2 $\mathbb{Z}_3$	NA 8 Ising	NA 2 $(SO(3), 5)$ BU	
A 5 Toric Code	A 4 $\mathbb{Z}_4$	NA 4 Fib × Semion BU	NA 2 $(SO(3), 7)$ BU	NA 3 DFib BU

The ith-row lists all rank $= i$ unitary modular tensor categories.

Middle symbol: the fusion rule.
Upper left corner: A = abelian theory, NA = non-abelian.
Upper right corner number = the number of distinct theories.
Lower left corner BU = there is a universal braiding anyon.

Realization as Topological Phase

Kitaev's Toric Code: $H = -\sum_v A_v - \sum_p B_p$



$= T^2$

L= $\otimes_{edges}\, \mathbb{C}^2$

A_v= $\otimes_{e \varepsilon v}\ \sigma^z\ \otimes_{others}\ Id_e,$

B_p= $\otimes_{e \varepsilon p}\ \sigma^x\ \otimes_{others}\ Id_e,$

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Toric Code Exactly Solvable

- A_v, B_p all commute with each other
- Ground states are $\cong \mathbb{C}^4$, i.e. 4-fold degenerate
- Gapped in the thermodynamic limit: $\lambda_1 - \lambda_0 > c > 0$
- Excitations are mutual anyons

Unitary Modular Category Realized by Toric Code

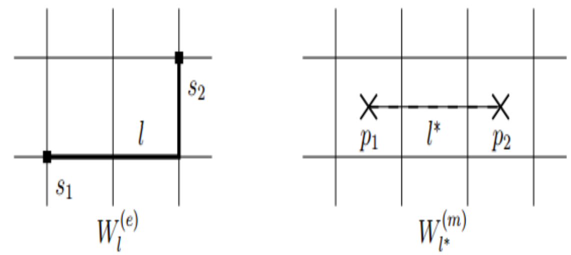
- 4 types of simple objects=anyons $\{1, e, m, \psi\}$:

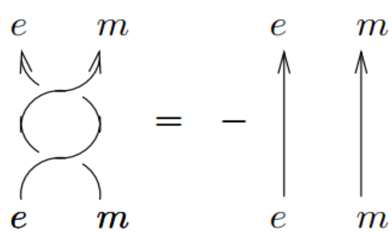
1=ground state or vaccum, e, m=bosons, ψ=fermion,

$e \otimes e = 1, m \otimes m = 1, e \otimes m = \psi$

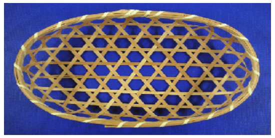
The fusion rule same as $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$.

- The anyons form a Drinfeld center $D(Z_2)$:





Herbertsmithite



Physical Theorem (Jiang-W.-Balents):

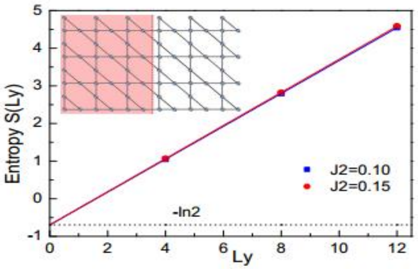
The spin=$\frac{1}{2}$ Heisenberg anti-ferromagnetic Kagome model

$$\mathbf{H}=J_1 \sum_{<ij>}(\sigma_i^x \sigma_j^x+\sigma_i^y \sigma_j^y+\sigma_i^z\sigma_j^z)+J_2 \sum_{\ll ij \gg}(\sigma_i^x \sigma_j^x+\sigma_i^y \sigma_j^y+\sigma_i^z\sigma_j^z)$$

represents a topological phase of matter which is in the same universality class of the **toric code** when $0<{}^{J_2}/_{J_1}<0.15$, where <ij> means summation over the nearest neighborhood and <<ij>> the next nearest neighborhood.

How to identify the MTC/TQFT?

Entanglement and classification of MTCs.



Entanglement

- **Relative to locality:**

Hilbert space of states $L = \bigotimes_{i \in s} L_i$ decomposed into parts L_i with $\dim L_i > 1$, a state ψ is a product if $\psi = \psi_\alpha \otimes \psi_\beta$ for some states $\psi_\alpha, \psi_\beta, \alpha \cup \beta = s$.

Otherwise, a state is entangled.

In quantum computation, $L_i = \mathbb{C}^2$-called a qubit.

Spin-singlet $\psi = (|01> - |10>)/\sqrt{2}$ entangled.

"Another way of expressing the peculiar situation is: the best possible knowledge of a *whole* does not necessarily include the best possible knowledge of its *parts* … I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought…

By the interaction the two representatives [quantum states] have become *entangled*."

- **Whole is more definite than parts:**

Spin-singlet ψ pure, but each qubit in a mixed state.

- **Spooky action at a distance:**

Measuring one results a definite state of the other.

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Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

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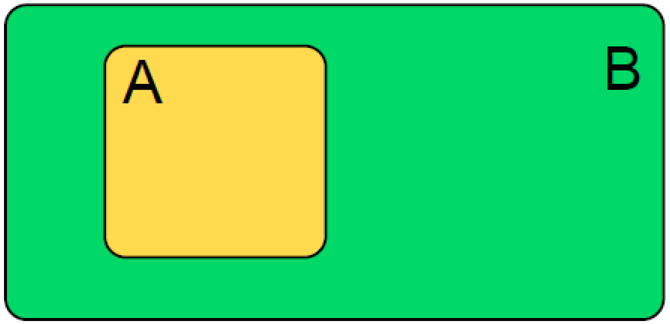
In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

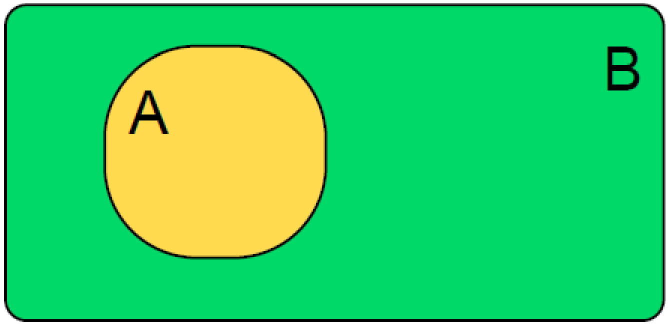
von Neumann Entropy



$$\rho_A = \mathrm{Tr}_B \, |\psi\rangle\langle\psi|$$

von Neumann $S_{vN}(A) = -\mathrm{Tr}\left[\rho_A \ln \rho_A\right]$

Topological Entanglement Entropy (TEE)



Preskill-Kitaev, Levin-Wen
2006

$$\boldsymbol{S_{vN} = \alpha L} \color{red}{\boldsymbol{-\gamma}} + \mathcal{O}(\frac{1}{L}),$$

$L =$ length of the smooth boundary.

TEE γ quantifies long-range entanglement.

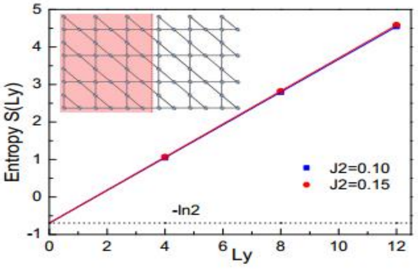
Identify MTC/TQFT

Compute topological entanglement entropy:

$\gamma = 0$, trivial.

$\gamma \neq 0$, some non-trivial MTC, $\boldsymbol{\gamma = \ln D}$.

But which one?



Use mathematical classification to identify:

Fix γ, only finitely many MTCs.

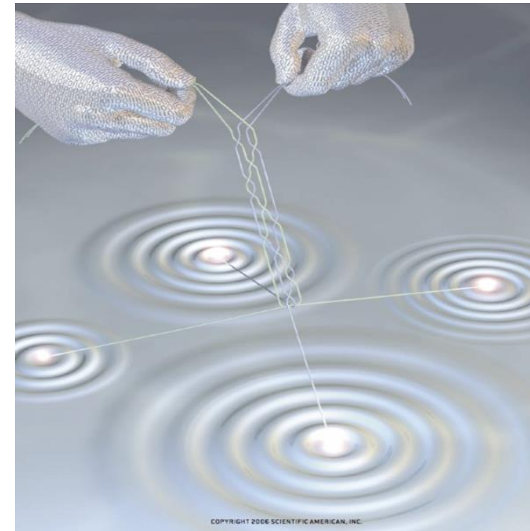
For simple cases, completely classified. In some cases, there are extra information to identify the MTC/TQFT, e.g. spin=$\frac{1}{2}$ Heisenberg anti-ferromagnetic Kagome model

Topological Quantum Computation

Topological quantum computation depends on

- the existence of non-Abelian topological phase

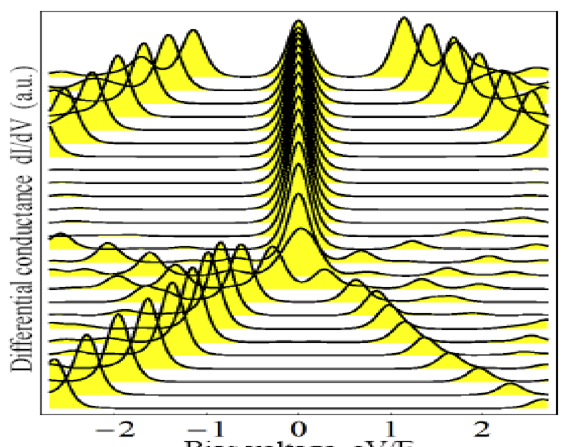
- the ability to manipulate quasiparticle excitations (anyons) in these phases



Candidate systems:

Fractional QH states

Topological nanowires



spin systems