This file contains the computations relevant to the paper "On the classification of non-self-dual modular categories" by Hong and Rowell.

```
> with(LinearAlgebra): with(Groebner):
```

Using the Galois argument assuming that (012)(34) is the generator of the group, one reduces (by excluding some possibilities or by permuting the labels) to the following S-matrix. We assume d,f,g are >1.

```
> S:=Matrix(5,5,[[1,d,f,g,g],[d,-f,-1,g,g],[f,-1,d,-g,-g],[g,g,-g,h1
  +I*h2,h1-I*h2],[g,g,-g,h1-I*h2,h1+I*h2]]);
```

$$S := \begin{bmatrix} 1 & d & f & g & g \\ d & -f & -1 & g & g \\ f & -1 & d & -g & -g \\ g & g & -g & h1+h2\,I & h1-h2\,I \\ g & g & -g & h1-h2\,I & h1+h2\,I \end{bmatrix}$$

```
> C:=Matrix(5,5,[[1,0,0,0,0],[0,1,0,0,0],[0,0,1,0,0],[0,0,0,0,1],[0,
  0,0,1,0]]);
```

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

These relations describe the condition that S^2 is proportiional to the "charge conjugation matrix."

```
> Srels:=factor(convert(evalm(S^2-K^2*C),set));
```

$$Srels := \{(1-f+2\,h1+d)\,g,\ -(1-f+2\,h1+d)\,g,\ 3\,g^2+2\,h1^2-2\,h2^2,\ d-d\,f-f+2\,g^2,\ f-d+d\,f-2\,g^2,\ 3\,g^2+2\,h1^2+2\,h2^2-K^2,\ 1+d^2+f^2+2\,g^2-K^2\}$$

```
> factor(Basis(‘union‘(Srels,{1+d-f+2*h1}),lexdeg([K,d,f,g],[h2,h1])
  ));
```

$$[1-f+2\,h1+d,\ 3\,g^2+2\,h1^2-2\,h2^2,\ -3\,f+3\,f^2-6\,f\,h1+3+6\,h1+4\,h1^2-4\,h2^2,\ (K-2\,h2)\,(K+2\,h2)]$$

We may assume h2>0 by conjugating if necessary.

```
> factor(Basis(‘union‘(Srels,{1+d-f+2*h1,K-2*h2}),tdeg(K,h2,h1,d,f,g
  )));
```

$$[1-f+2\,h1+d,\ K-2\,h2,\ f-d+d\,f-2\,g^2,\ 4\,h2^2-1-d^2-f^2-2\,g^2]$$

These relations come down to a single relation among the d,f and g (the other allow elimination of K, h1 and h2).

```
> orthrel:=-d+d*f+f-2*g^2;
```

$$orthrel := f-d+d\,f-2\,g^2$$

We use the various symmetries of the N_{i,j}^k to write down the fusion matrices in terms of just 14 variables. Note that M4 is just the transpose of M3.

```
> M1:=Matrix([[0,1,0,0,0],[1,n1,n2,n3,n3],[0,n2,n4,n5,n5],[0,n3,n5,n
```

```
6,n7],[0,n3,n5,n7,n6]]);
```

$$M1 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & n1 & n2 & n3 & n3 \\ 0 & n2 & n4 & n5 & n5 \\ 0 & n3 & n5 & n6 & n7 \\ 0 & n3 & n5 & n7 & n6 \end{bmatrix}$$

```
> M2:=Matrix([[0,0,1,0,0],[0,n2,n4,n5,n5],[1,n4,n8,n9,n9],[0,n5,n9,n
  10,n11],[0,n5,n9,n11,n10]]);
```

$$M2 := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & n2 & n4 & n5 & n5 \\ 1 & n4 & n8 & n9 & n9 \\ 0 & n5 & n9 & n10 & n11 \\ 0 & n5 & n9 & n11 & n10 \end{bmatrix}$$

```
> M3:=Matrix([[0,0,0,0,1],[0,n3,n5,n7,n6],[0,n5,n9,n11,n10],[1,n6,n1
  0,n12,n13],[0,n7,n11,n14,n12]]);
```

$$M3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & n3 & n5 & n7 & n6 \\ 0 & n5 & n9 & n11 & n10 \\ 1 & n6 & n10 & n12 & n13 \\ 0 & n7 & n11 & n14 & n12 \end{bmatrix}$$

The matrices must commute, giving a set of diophantine equations.

```
> comrels:=`minus`(`union`(convert(evalm(M1&*M2-M2&*M1),set),convert
  (evalm(M1&*M3-M3&*M1),set),convert(evalm(M2&*M3-M3&*M2),set),conve
  rt(evalm(M3&*Transpose(M3)-Transpose(M3)&*M3),set)),{0});
```

$comrels := \{n12 - n13,\ n13 - n12,\ n3\ n6 + n5\ n10 + n7\ n13 - n3\ n7 - n5\ n11 - n7\ n14,$

$n3\ n7 + n5\ n11 + n7\ n14 - n3\ n6 - n5\ n10 - n7\ n13,$

$n6\ n5 + n9\ n10 + n11\ n13 - n7\ n5 - n9\ n11 - n11\ n14,$

$n7\ n5 + n9\ n11 + n11\ n14 - n6\ n5 - n9\ n10 - n11\ n13,\ 1 + n6^2 + n10^2 + n13^2 - n7^2 - n11^2 - n14^2,$

$1 + n1\ n4 + n2\ n8 + 2\ n3\ n9 - n2^2 - n4^2 - 2\ n5^2,\ n2^2 + n4^2 + 2\ n5^2 - 1 - n1\ n4 - n2\ n8 - 2\ n3\ n9,$

$n3^2 + n5^2 + 2\ n7\ n6 - n1\ n7 - n2\ n11 - n3\ n12 - n3\ n14,$

$n5^2 + n9^2 + 2\ n11\ n10 - n4\ n7 - n8\ n11 - n9\ n12 - n9\ n14,$

$n7^2 + n11^2 + n14^2 - 1 - n6^2 - n10^2 - n13^2,$

$n1\ n7 + n2\ n11 + n3\ n12 + n3\ n14 - n3^2 - n5^2 - 2\ n7\ n6,$

$$n4\,n7+n8\,n11+n9\,n12+n9\,n14-n5^2-n9^2-2\,n11\,n10,$$
$$n1\,n5+n2\,n9+n3\,n10+n3\,n11-n3\,n2-n5\,n4-n6\,n5-n7\,n5,$$
$$n2\,n5+n4\,n9+n5\,n10+n5\,n11-n3\,n4-n5\,n8-n6\,n9-n7\,n9,$$
$$n2\,n6+n4\,n10+n5\,n13+n5\,n12-n3\,n5-n5\,n9-n6\,n10-n7\,n11,$$
$$n2\,n7+n4\,n11+n5\,n12+n5\,n14-n3\,n5-n5\,n9-n6\,n11-n7\,n10,$$
$$n3\,n2+n5\,n4+n6\,n5+n7\,n5-n1\,n5-n2\,n9-n3\,n10-n3\,n11,$$
$$n3\,n4+n5\,n8+n6\,n9+n7\,n9-n2\,n5-n4\,n9-n5\,n10-n5\,n11,$$
$$n3\,n5+n5\,n9+n6\,n10+n7\,n11-n2\,n6-n4\,n10-n5\,n13-n5\,n12,$$
$$n3\,n5+n5\,n9+n6\,n11+n7\,n10-n2\,n7-n4\,n11-n5\,n12-n5\,n14,$$
$$n3\,n6+n5\,n10+n6\,n13+n7\,n12-n3\,n7-n5\,n11-n6\,n12-n7\,n14,$$
$$n3\,n7+n5\,n11+n6\,n12+n7\,n14-n3\,n6-n5\,n10-n6\,n13-n7\,n12,$$
$$n6\,n5+n9\,n10+n10\,n13+n11\,n12-n7\,n5-n9\,n11-n10\,n12-n11\,n14,$$
$$n7\,n5+n9\,n11+n10\,n12+n11\,n14-n6\,n5-n9\,n10-n10\,n13-n11\,n12,$$
$$1+n1\,n6+n2\,n10+n3\,n13+n3\,n12-n3^2-n5^2-n6^2-n7^2,$$
$$1+n4\,n6+n8\,n10+n9\,n13+n9\,n12-n5^2-n9^2-n10^2-n11^2,$$
$$n3^2+n5^2+n6^2+n7^2-1-n1\,n6-n2\,n10-n3\,n13-n3\,n12,$$
$$n5^2+n9^2+n10^2+n11^2-1-n4\,n6-n8\,n10-n9\,n13-n9\,n12\}$$

```
> indets(comrels);
```

$$\{n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9\}$$

The commutation relations alone leave 6 degrees of freedom.

```
> HilbertDimension(comrels,tdeg(n1, n10, n11, n12, n13, n14, n2, n3,
  n4, n5, n6, n7, n8, n9));
```

$$6$$

The Characteristic polynomials of M1,M2 and M3 are computed in two ways. First from the knowledge of the eigenvalues.

```
> ch1:=collect(expand(product('(X-S[2,i]/S[1,i])',i=1..5)),X);
  ch2:=collect(expand(product('(X-S[3,i]/S[1,i])',i=1..5)),X);
  ch3:=collect(expand(product('(X-S[4,i]/S[1,i])',i=1..5)),X);
```

$$ch1 := -1+X^5+\left(-d+\frac{1}{f}-2+\frac{f}{d}\right)X^4+\left(2\,d+1-\frac{2\,f}{d}-f-\frac{2}{f}+\frac{1}{d}-\frac{d}{f}\right)X^3$$
$$+\left(\frac{f}{d}-d+2\,f+\frac{1}{f}+\frac{2\,d}{f}-1-\frac{2}{d}\right)X^2+\left(-\frac{d}{f}+2-f+\frac{1}{d}\right)X$$

$$ch2 := 1+X^5+\left(-\frac{d}{f}+2-f+\frac{1}{d}\right)X^4+\left(-2\,f+1+\frac{2}{d}-\frac{f}{d}-\frac{2\,d}{f}-\frac{1}{f}+d\right)X^3$$
$$+\left(2\,d+1-\frac{2\,f}{d}-f-\frac{2}{f}+\frac{1}{d}-\frac{d}{f}\right)X^2+\left(d+2-\frac{f}{d}-\frac{1}{f}\right)X$$

$$ch3 := X^5+\left(-\frac{2\,h1}{g}+\frac{g}{f}-\frac{g}{d}-g\right)X^4+\left(-\frac{2\,h1}{f}+\frac{g^2}{d}+\frac{2\,h1}{d}+\frac{h2^2}{g^2}-\frac{g^2}{d\,f}+\frac{h1^2}{g^2}-\frac{g^2}{f}+2\,h1\right)X^3$$

$$+\left(\frac{h2^2}{f\,g}+\frac{g^3}{d\,f}+\frac{2\,g\,h1}{d\,f}+\frac{h1^2}{f\,g}-\frac{h2^2}{d\,g}+\frac{2\,g\,h1}{f}-\frac{h1^2}{d\,g}-\frac{2\,g\,h1}{d}-\frac{h2^2}{g}-\frac{h1^2}{g}\right)X^2$$

$$+\left(\frac{h1^2}{d}-\frac{h1^2}{d\,f}-\frac{2\,g^2\,h1}{d\,f}+\frac{h2^2}{d}-\frac{h1^2}{f}-\frac{h2^2}{d\,f}-\frac{h2^2}{f}\right)X+\frac{g\,h2^2}{d\,f}+\frac{g\,h1^2}{d\,f}$$

The form of these polynomials imply further relations.

```
> {coeff(ch1,X,0)+1,coeff(ch2,X,0)-1,coeff(ch1,X,4)+coeff(ch2,X,1),c
  oeff(ch1,X,1)-coeff(ch2,X,4),coeff(ch1,X,3)-coeff(ch2,X,2),coeff(c
  h1,X,2)+coeff(ch2,X,3)};
```

$$\{0\}$$

```
> factor(ch1); factor(ch2); factor(ch3);
```

$$\frac{(X-1)^2\,(f\,X+1)\,(X-d)\,(d\,X+f)}{d\,f}$$

$$\frac{(X+1)^2\,(X-f)\,(d\,X+1)\,(f\,X-d)}{d\,f}$$

$$\frac{(X-g)\,(g^2\,X^2-2\,g\,h1\,X+h1^2+h2^2)\,(f\,X+g)\,(d\,X-g)}{g^2\,f\,d}$$

```
> p1:=CharacteristicPolynomial(M1,X):
  p2:=CharacteristicPolynomial(M2,X):
  p3:=CharacteristicPolynomial(M3,X):
> factor(p1); factor(p2);
```

$$(n7+X-n6)\,(X^2\,n1\,n4+2\,n5^2-n4\,n7-n4\,n6-X^3\,n6-X^3\,n4+X^2\,n6\,n4-2\,X^2\,n5^2+X^4-X^2$$
$$+X\,n7+X\,n6-4\,X\,n2\,n3\,n5-X\,n6\,n1\,n4-X\,n1\,n4\,n7-X^3\,n1-2\,X^2\,n3^2-X^2\,n2^2+X\,n4$$
$$-X^3\,n7+X^2\,n1\,n6+X^2\,n1\,n7+X^2\,n4\,n7+2\,X\,n1\,n5^2+2\,X\,n3^2\,n4+X\,n6\,n2^2+X\,n2^2\,n7)$$

$$(X-n10+n11)\,(X^2\,n2\,n8+2\,n5^2-n2\,n11-n2\,n10-2\,X^2\,n5^2+X^2\,n2\,n10+X^4-X^2-X^3\,n8$$
$$-X^3\,n10+X\,n10-2\,X^2\,n9^2+X\,n11-X\,n10\,n2\,n8-4\,X\,n4\,n5\,n9-X\,n2\,n8\,n11-X^3\,n2$$
$$-X^2\,n4^2+X\,n2-X^3\,n11+X^2\,n10\,n8+X^2\,n8\,n11+X^2\,n2\,n11+2\,X\,n5^2\,n8+2\,X\,n2\,n9^2$$
$$+X\,n10\,n4^2+X\,n4^2\,n11)$$

Observe that the only linear term of p1 and p2 are (X-1) and (X+1) respectively.

```
> solve(-n10+n11+X=(X+1));
```

$$\{X=X,\,n10=n11-1,\,n11=n11\}$$

```
> solve(n7+X-n6=(X-1));
```

$$\{X=X,\,n6=n7+1,\,n7=n7\}$$

```
> linrels:={n11-(n10+1),n6-(n7+1)};
```

$$linrels:=\{-1-n7+n6,\,-n10+n11-1\}$$

The relations implied by the coefficients of the Characteristic polynomials.

```
> chrels:={coeff(p1,X,0)+1,coeff(p2,X,0)-1,coeff(p1,X,4)+coeff(p2,X,
  1),coeff(p1,X,1)-coeff(p2,X,4),coeff(p1,X,3)-coeff(p2,X,2),coeff(p
  1,X,2)+coeff(p2,X,3)};
```

$$chrels:=\{2\,n11\,n5^2-2\,n10\,n5^2-n11^2\,n2+n2\,n10^2-1,$$

$n4\ n6^2 - n4\ n7^2 - 2\ n5^2\ n6 + 2\ n5^2\ n7 + 1, -2\ n6 - n1 - n4 - 4\ n5\ n4\ n9\ n11 + n10^2\ n2\ n8$
$- 2\ n5^2\ n8\ n10 - 2\ n9^2\ n2\ n10 - 2\ n2\ n10 + 2\ n9^2\ n11\ n2 + 4\ n10\ n4\ n5\ n9 + 2\ n11\ n5^2\ n8$
$- n11^2\ n2\ n8 + 2\ n5^2 + n11^2\ n4^2 - n10^2 + n11^2 - n10^2\ n4^2, 4\ n2\ n3\ n5\ n6 - 2\ n3^2\ n4\ n6$
$- n1\ n4\ n7^2 - 4\ n2\ n3\ n5\ n7 + n1\ n4\ n6^2 + 2\ n1\ n5^2\ n7 + 2\ n5^2 - 2\ n1\ n5^2\ n6 + n7^2 + 2\ n3^2\ n7\ n4$
$- n6^2 + n7^2\ n2^2 - n6^2\ n2^2 - 2\ n4\ n6 + n2 + 2\ n10 + n8, -2\ n3^2 + n1\ n4 - n2^2 + 2\ n1\ n6 + 2\ n4\ n6$
$- n7^2 - 1 - 2\ n5^2 + n6^2 - n11^2\ n2 + 2\ n10\ n2\ n8 - 2\ n10\ n5^2 + 4\ n4\ n5\ n9 - n11^2\ n8 + 2\ n9^2\ n11$
$- 2\ n5^2\ n8 + n2\ n10^2 - 2\ n10 - 2\ n2\ n9^2 - 2\ n10\ n4^2 - n2 - 2\ n10\ n9^2 + n8\ n10^2 + 2\ n11\ n5^2,$
$2\ n6\ n2^2 - 2\ n3^2\ n7 - 2\ n5^2\ n7 + 2\ n3^2\ n4 - n1\ n6^2 + n1\ n7^2 + 2\ n1\ n5^2 - n4\ n6^2 + 2\ n5^2\ n6$
$- 4\ n2\ n3\ n5 - 2\ n6\ n1\ n4 + 2\ n6 + 2\ n3^2\ n6 + n4 + n4\ n7^2 - 1 - n4^2 + 2\ n2\ n10 - 2\ n9^2 + n2\ n8$
$+ 2\ n8\ n10 - 2\ n5^2 - n11^2 + n10^2 \}$

```
> nops(chrels);
```

6

We take all of the relations together and process them. We rename three of the variables u,v,t; for later use.

```
> allNrels:=subs({n10=t,n12=v,n14=u},`union`(linrels,comrels,chrels)
  ):
> factor(Basis(allNrels,plex(n1, n11, n13, n2, n3, n4, n5, n6, n7,
  n8, n9,t,u,v)));
```

$[(75 + 8\ v^2)\ (-v + u)\ (2\ u^2\ t - u^2 - 4\ u\ v - 3\ v^2 - 2\ t\ v^2 + 2\ t + 1 + 4\ t^2),$
$(5\ u - v)\ (-v + u)\ (2\ u^2\ t - u^2 - 4\ u\ v - 3\ v^2 - 2\ t\ v^2 + 2\ t + 1 + 4\ t^2),$
$(-v + u)\ (2\ t + 14 - u^2 + v^2)\ (2\ u^2\ t - u^2 - 4\ u\ v - 3\ v^2 - 2\ t\ v^2 + 2\ t + 1 + 4\ t^2),$
$(2\ u^2\ t - u^2 - 4\ u\ v - 3\ v^2 - 2\ t\ v^2 + 2\ t + 1 + 4\ t^2)$
$(20\ t^2 + 10\ t - 10\ u^2\ t + 10\ t\ v^2 + 5 - 104\ u\ v + 99\ v^2 + 5\ u^2 + 5\ u^4 - 10\ u^2\ v^2 + 5\ v^4), -5 + 5\ u^2$
$- 3\ v^4 - 10\ t - 24\ v^2 + 18\ v\ u\ t - v^3\ u - 8\ t\ v^2 + u^3\ v - 10\ u^2\ t + 19\ u\ v + 20\ n9\ v - 20\ t^2 + 3\ u^2\ v^2$
$- 2\ v^4\ t + 2\ v^3\ u\ t - 2\ u^3\ v\ t - 4\ u\ v\ t^2 + 4\ v^2\ t^2 + 2\ u^2\ t\ v^2, 15\ u^2 + 3\ v^4 - 30\ t + 44\ v^2 - 58\ v\ u\ t$
$+ v^3\ u + 28\ t\ v^2 - u^3\ v + 30\ u^2\ t - 59\ u\ v + 60\ n9\ u - 60\ t^2 - 3\ u^2\ v^2 - 15 + 2\ v^4\ t - 2\ v^3\ u\ t + 2\ u^3\ v\ t$
$+ 4\ u\ v\ t^2 - 4\ v^2\ t^2 - 2\ u^2\ t\ v^2, -103\ v^3 + 37\ u^3 + 4\ n9 + 27\ v - 35\ u + 54\ v\ t - 70\ u\ t - 37\ v^2\ u$
$+ 103\ u^2\ v + 62\ v^2\ u\ t + 108\ v\ t^2 + 12\ v^3\ u^2 - 6\ u^4\ v - 2\ u^5 + 16\ n9\ t^2 + 78\ v\ u^2\ t - 2\ u\ v^4 + 4\ u^3\ v^2$
$+ 8\ n9\ t - 8\ u^3\ t\ v^2 - 4\ u^4\ t\ v + 4\ u^5\ t - 4\ v^5\ t - 78\ v^3\ t + 8\ u^2\ t\ v^3 + 4\ v^4\ u\ t - 62\ u^3\ t - 140\ u\ t^2$
$- 6\ v^5, -6 - 31\ v^2 + 22\ u\ v + 9\ u^2 - 12\ t + 3\ v^4 + 10\ v^3\ u - 38\ t\ v^2 - 10\ u^3\ v + 44\ v\ u\ t - 3\ u^4 - 6\ u^2\ t$
$- 24\ t^2 + 2\ v^4\ t + 4\ v^3\ u\ t - 8\ u^2\ t\ v^2 - 16\ v^2\ t^2 - 4\ u^3\ v\ t + 16\ u\ v\ t^2 + 6\ u^4\ t + 12\ n9^2, 5 + 10\ t$
$- 68\ v^2 + 28\ u\ v - 151\ t\ v^2 + 66\ v\ u\ t + 5\ u^2\ t + 34\ v^4 + 118\ v^3\ u - 114\ u^2\ v^2 - 38\ u^3\ v - 152\ v^2\ t^2$
$+ 152\ u\ v\ t^2 + 36\ v^4\ t + 4\ v^3\ u\ t - 116\ u^2\ t\ v^2 + 76\ u^3\ v\ t - 5\ n8 + 40\ v^2\ n8, -6\ v + 2\ u - 12\ v\ t + 4\ u\ t$
$+ 3\ v^3 + 7\ v^2\ u - 7\ u^2\ v - 3\ u^3 - 12\ v\ t^2 + 12\ u\ t^2 + 3\ v^3\ t - v^2\ u\ t - 7\ v\ u^2\ t + 5\ u^3\ t + 3\ v\ n8 + n8\ u,$
$-10 - 23\ v^2 + 18\ u\ v + 5\ u^2 - 40\ t + 9\ v^4 + 18\ v^3\ u - 4\ u^2\ v^2 - 56\ t\ v^2 - 18\ u^3\ v + 56\ v\ u\ t - 5\ u^4$

$- 40\, t^2 + 6\, v^4\, t + 4\, v^3\, u\, t - 16\, u^2\, t\, v^2 - 32\, v^2\, t^2 - 4\, u^3\, v\, t + 32\, u\, v\, t^2 + 10\, u^4\, t + 10\, n8 + 20\, n8\, t,$
$-42\, v^3 + 8\, u^3 - 4\, n9 + 9\, v - 9\, u + 18\, v\, t - 18\, u\, t + 4\, v^2\, u + 30\, u^2\, v + 44\, v^2\, u\, t + 4\, n8\, n9 + 36\, v\, t^2$
$+ 2\, v^3\, u^2 + u^4\, v + u^5 + 8\, v\, u^2\, t + 5\, u\, v^4 - 6\, u^3\, v^2 - 8\, n9\, t - 4\, u^3\, t\, v^2 + 6\, u^4\, t\, v - 2\, u^5\, t - 2\, v^5\, t$
$- 32\, v^3\, t - 4\, u^2\, t\, v^3 + 6\, v^4\, u\, t - 20\, u^3\, t - 36\, u\, t^2 - 3\, v^5,\ 1 + v^2 - u^2 + 2\, n7 - 2\, t,$
$-1 - 2\, t + v^2 - u^2 + 2\, n6,\ v\, t - u\, t + n5 - n9 + v - u,\ 29\, v^2 - 29\, u\, v + 13\, t\, v^2 - 28\, v\, u\, t + 15\, u^2\, t$
$+ 3\, v^4 + v^3\, u - 3\, u^2\, v^2 - u^3\, v - 4\, v^2\, t^2 + 4\, u\, v\, t^2 + 2\, v^4\, t - 2\, v^3\, u\, t - 2\, u^2\, t\, v^2 + 2\, u^3\, v\, t - 15\, n8$
$+ 15\, n4,\ v - u - 2\, n9 + 4\, v\, t - 4\, u\, t - v^3 + v^2\, u + u^2\, v - u^3 + 2\, n3,\ 15 + 43\, v^2 - 28\, u\, v - 15\, u^2$
$+ 11\, t\, v^2 - 26\, v\, u\, t + 15\, u^2\, t + 6\, v^4 + 2\, v^3\, u - 6\, u^2\, v^2 - 2\, u^3\, v - 8\, v^2\, t^2 + 8\, u\, v\, t^2 + 4\, v^4\, t - 4\, v^3\, u\, t$
$- 4\, u^2\, t\, v^2 + 4\, u^3\, v\, t - 15\, n8 + 15\, n2,\ n13 - v,\ -t + n11 - 1,\ 10 + 43\, v^2 - 28\, u\, v - 15\, u^2 - 4\, t\, v^2$
$+ 4\, v\, u\, t + 11\, v^4 - 8\, v^3\, u - 6\, u^2\, v^2 + 8\, u^3\, v - 5\, u^4 - 8\, v^2\, t^2 + 8\, u\, v\, t^2 + 4\, v^4\, t - 4\, v^3\, u\, t - 4\, u^2\, t\, v^2$
$+ 4\, u^3\, v\, t - 10\, n8 + 10\, n1]$

The first relation implies that either dio1 or bad1 is zero. We will see that bad1 cannot be 0.

```
> dio1:=4*t^2+2*u^2*t+2*t-2*t*v^2-u^2-4*u*v+1-3*v^2;
> bad1:=u-v;
```

$$dio1 := 2\, u^2\, t - u^2 - 4\, u\, v - 3\, v^2 - 2\, t\, v^2 + 2\, t + 1 + 4\, t^2$$

$$bad1 := -v + u$$

```
> factor(Basis(`union`(allNrels,{bad1}),plex(n1, n11, n13, n2, n3,
  n4, n5, n6, n7, n8, n9,t,u,v)));
```

$[-v + u,\ (2\, t + 1 + 4\, t^2)\, (4\, t^2 + 2\, t + 1 - 8\, v^2),\ -1 + 4\, n9\, v - 2\, t - 4\, t^2,\ (2\, t + 1 + 4\, t^2)\, (n9 - 2\, v),$
$-1 + 2\, n9^2 - 4\, t^2 - 2\, t,\ n8 - 1 - 2\, t,\ 2\, n7 + 1 - 2\, t,\ 2\, n6 - 1 - 2\, t,\ -n9 + n5,\ n4 - 1 - 2\, t,\ n3 - n9,$
$n2 - 2\, t,\ n13 - v,\ -t + n11 - 1,\ n1 - 2\, t]$

```
> bad2:=1-8*v^2+2*t+4*t^2;
```

$$bad2 := 4\, t^2 + 2\, t + 1 - 8\, v^2$$

```
> `mod`(bad2,4);
```

$$2\, t + 1$$

```
> factor(Basis(`union`(allNrels,{dio1}),plex(n1, n11, n13, n2, n3,
  n4, n5, n6, n7, n8, n9,t,u,v)));
```

$[2\, u^2\, t - u^2 - 4\, u\, v - 3\, v^2 - 2\, t\, v^2 + 2\, t + 1 + 4\, t^2,\ v\, (-2\, v - v\, t + n9 + u\, t),\ u\, (-2\, v - v\, t + n9 + u\, t),\ u^2$
$- 3\, v^4 - u^4 - 15\, v^2 + 12\, v\, u\, t + 2\, v^3\, u - 14\, t\, v^2 - 2\, u^3\, v + 2\, u^2\, t - 2\, u\, v + 4\, n9^2 + 2\, u^4\, t + 4\, u^2\, v^2$
$- 2\, v^4\, t + 4\, v^3\, u\, t - 4\, u^3\, v\, t,\ (8\, v^2 - 1)\, (-2\, v^2 - t\, v^2 + 2\, u\, v + 2\, v\, u\, t + n8 - 2\, t - 1 - u^2\, t),$
$(3\, v + u)\, (-2\, v^2 - t\, v^2 + 2\, u\, v + 2\, v\, u\, t + n8 - 2\, t - 1 - u^2\, t),\ -9\, v^2 - 6\, u\, v - u^2 - 4\, t - 3\, v^4 + 2\, v^3\, u$
$+ 4\, u^2\, v^2 - 12\, t\, v^2 - 2\, u^3\, v + 8\, v\, u\, t - u^4 + 4\, u^2\, t - 2\, v^4\, t + 4\, v^3\, u\, t - 4\, u^3\, v\, t + 2\, u^4\, t + 2\, n8 + 4\, n8\, t,$
$-15\, v^3 + 13\, v^2\, u + 3\, u^2\, v - u^3 - 3\, v^5 + 5\, u\, v^4 + 2\, v^3\, u^2 - 14\, v^3\, t - 6\, u^3\, v^2 + 26\, v^2\, u\, t + u^4\, v$
$- 10\, v\, u^2\, t + u^5 - 2\, u^3\, t - 2\, v^5\, t + 6\, v^4\, u\, t - 4\, u^2\, t\, v^3 - 4\, u^3\, t\, v^2 + 6\, u^4\, t\, v - 2\, u^5\, t - 4\, n9 - 8\, n9\, t$
$+ 4\, n8\, n9,\ 1 + v^2 - u^2 + 2\, n7 - 2\, t,\ -1 - 2\, t + v^2 - u^2 + 2\, n6,\ v\, t - u\, t + n5 - n9 + v - u,$
$2\, v^2 - 2\, u\, v + t\, v^2 - 2\, v\, u\, t + u^2\, t - n8 + n4,\ v - u - 2\, n9 + 4\, v\, t - 4\, u\, t - v^3 + v^2\, u + u^2\, v - u^3 + 2\, n3,$

$1+3\,v^2-2\,u\,v-u^2+t\,v^2-2\,v\,u\,t+u^2\,t-n8+n2,\ n13-v,\ -t+n11-1,$

$2+9\,v^2-6\,u\,v-3\,u^2-2\,n8+v^4-2\,v^3\,u+2\,u^3\,v-u^4+2\,n1]$

Since (8v^2-1) cannot be 0, get a new relation.

```
> newrel1:=-t*v^2-2*v^2+2*u*v+2*v*u*t-1-2*t-u^2*t+n8;newrel1 :=
  -2*v^2-t*v^2+2*u*v+2*v*u*t+n8-2*t-1-u^2*t
> factor(Basis(‘union‘(allNrels,{dio1,newrel1}),plex(n1, n11, n13,
  n2, n3, n4, n5, n6, n7, n8, n9,t,u,v)));
```

$[2\,u^2\,t-u^2-4\,u\,v-3\,v^2-2\,t\,v^2+2\,t+1+4\,t^2,\ v\,(-2\,v-v\,t+n9+u\,t),\ u\,(-2\,v-v\,t+n9+u\,t),\ u^2$
$-3\,v^4-u^4-15\,v^2+12\,v\,u\,t+2\,v^3\,u-14\,t\,v^2-2\,u^3\,v+2\,u^2\,t-2\,u\,v+4\,n9^2+2\,u^4\,t+4\,u^2\,v^2$
$-2\,v^4\,t+4\,v^3\,u\,t-4\,u^3\,v\,t,\ -2\,v^2-t\,v^2+2\,u\,v+2\,v\,u\,t+n8-2\,t-1-u^2\,t,\ 1+v^2-u^2+2\,n7-2\,t,$
$-1-2\,t+v^2-u^2+2\,n6,\ v\,t-u\,t+n5-n9+v-u,\ n4-1-2\,t,$
$v-u-2\,n9+4\,v\,t-4\,u\,t-v^3+v^2\,u+u^2\,v-u^3+2\,n3,\ -2\,t+v^2-u^2+n2,\ n13-v,\ -t+n11-1,$
$5\,v^2-2\,u\,v-3\,u^2-4\,t+v^4-2\,v^3\,u-2\,t\,v^2+2\,u^3\,v+4\,v\,u\,t-u^4-2\,u^2\,t+2\,n1]$

```
> goodrel1:=-v*t-2*v+n9+u*t;
```

$$goodrel1 := -2\,v-v\,t+n9+u\,t$$

Either goodrel1=0 or both u and v are zero, by the second and third equations.

```
> badrels1:={u,v};
```

$$badrels1 := \{u,\ v\}$$

No integer solutions for t with badrels1.

```
> factor(Basis(‘union‘(allNrels,{dio1,newrel1},badrels1),plex(n1,
  n11, n13, n2, n3, n4, n5, n6, n7, n8, n9,t,u,v)));
```

$[v,\ u,\ 4\,t^2+2\,t+1,\ n9^2,\ n8-1-2\,t,\ 2\,n7+1-2\,t,\ 2\,n6-1-2\,t,\ -n9+n5,\ n4-1-2\,t,\ n3-n9,$
$n2-2\,t,\ n13,\ -t+n11-1,\ n1-2\,t]$

```
> finalNs:=factor(Basis(‘union‘(allNrels,{dio1,newrel1,goodrel1}),pl
  ex(n1, n11, n13, n2, n3, n4, n5, n6, n7, n8, n9,t,u,v)));
```

$finalNs := [2\,u^2\,t-u^2-4\,u\,v-3\,v^2-2\,t\,v^2+2\,t+1+4\,t^2,\ -2\,v-v\,t+n9+u\,t,$
$-2\,v^2-t\,v^2+2\,u\,v+2\,v\,u\,t+n8-2\,t-1-u^2\,t,\ 1+v^2-u^2+2\,n7-2\,t,\ -1-2\,t+v^2-u^2+2\,n6,$
$-v-u+n5,\ n4-1-2\,t,\ -3\,v-u+2\,v\,t-2\,u\,t-v^3+v^2\,u+u^2\,v-u^3+2\,n3,\ -2\,t+v^2-u^2+n2,$
$n13-v,\ -t+n11-1,$
$5\,v^2-2\,u\,v-3\,u^2-4\,t+v^4-2\,v^3\,u-2\,t\,v^2+2\,u^3\,v+4\,v\,u\,t-u^4-2\,u^2\,t+2\,n1]$

We obtain one diophanitine equation in t,u,v and the remaining fusion coefficients can be expressed uniquely as polynomials in t,u,v.

```
> rules1:=‘union‘({n10=t,n12=v,n14=u},solve({seq(finalNs[i],i=2..nop
  s(finalNs))},{n1, n11, n13, n2, n3, n4, n5, n6, n7, n8, n9}));
```

$rules1 := \{n1=-\frac{5}{2}\,v^2+u\,v+\frac{3}{2}\,u^2+2\,t-\frac{1}{2}\,v^4+v^3\,u+t\,v^2-u^3\,v-2\,v\,u\,t+\frac{1}{2}\,u^4+u^2\,t,\ n10=t,$
$n11=1+t,\ n12=v,\ n13=v,\ n14=u,\ n2=2\,t-v^2+u^2,$
$n3=\frac{3}{2}\,v+\frac{1}{2}\,u-v\,t+u\,t+\frac{1}{2}\,v^3-\frac{1}{2}\,v^2\,u-\frac{1}{2}\,u^2\,v+\frac{1}{2}\,u^3,\ n4=2\,t+1,\ n5=u+v,\ n6=\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2},$

$n7=-\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}$, $n8=t\,v^2+2\,v^2-2\,u\,v-2\,v\,u\,t+1+2\,t+u^2\,t$, $n9=v\,t+2\,v-u\,t\}$

The anticipated unique solution corresponding to the known example of SU(3)_4/Z_3.

```
> subs({t=0,v=0,u=1},%);
```

$\{n1=2, n10=0, n11=1, n12=0, n13=0, n14=1, n2=1, n3=1, n4=1, n5=1, n6=1, n7=0,$
$n8=1, n9=0\}$

The fusion matrices with all but t,u,v eliminated.

```
> M1s:=subs(rules1,M1); M2s:=subs(rules1,M2); M3s:=subs(rules1,M3);
```

$M1s :=$

$[0,1,0,0,0]$

$\left[1, -\frac{5}{2}v^2+u\,v+\frac{3}{2}u^2+2\,t-\frac{1}{2}v^4+v^3\,u+t\,v^2-u^3\,v-2\,v\,u\,t+\frac{1}{2}u^4+u^2\,t, 2\,t-v^2+u^2,\right.$
$\frac{3}{2}v+\frac{1}{2}u-v\,t+u\,t+\frac{1}{2}v^3-\frac{1}{2}v^2\,u-\frac{1}{2}u^2\,v+\frac{1}{2}u^3, \frac{3}{2}v+\frac{1}{2}u-v\,t+u\,t+\frac{1}{2}v^3-\frac{1}{2}v^2\,u-\frac{1}{2}u^2\,v+\frac{1}{2}u^3$
$\left.\right]$

$[0, 2\,t-v^2+u^2, 2\,t+1, u+v, u+v]$

$\left[0, \frac{3}{2}v+\frac{1}{2}u-v\,t+u\,t+\frac{1}{2}v^3-\frac{1}{2}v^2\,u-\frac{1}{2}u^2\,v+\frac{1}{2}u^3, u+v, \frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}, -\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right]$

$\left[0, \frac{3}{2}v+\frac{1}{2}u-v\,t+u\,t+\frac{1}{2}v^3-\frac{1}{2}v^2\,u-\frac{1}{2}u^2\,v+\frac{1}{2}u^3, u+v, -\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}, \frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right]$

$$M2s := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2\,t-v^2+u^2 & 2\,t+1 & u+v & u+v \\ 1 & 2\,t+1 & t\,v^2+2\,v^2-2\,u\,v-2\,v\,u\,t+1+2\,t+u^2\,t & v\,t+2\,v-u\,t & v\,t+2\,v-u\,t \\ 0 & u+v & v\,t+2\,v-u\,t & t & 1+t \\ 0 & u+v & v\,t+2\,v-u\,t & 1+t & t \end{bmatrix}$$

$M3s :=$

$[0,0,0,0,1]$

$\left[0, \frac{3}{2}v+\frac{1}{2}u-v\,t+u\,t+\frac{1}{2}v^3-\frac{1}{2}v^2\,u-\frac{1}{2}u^2\,v+\frac{1}{2}u^3, u+v, -\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}, \frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right]$

$[0, u+v, v\,t+2\,v-u\,t, 1+t, t]$

$\left[1, \frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}, t, v, v\right]$

$$\left[0\,,-\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\,,1+t\,,u\,,v\right]$$

The unique positive character is psi0.

```
> psi0:=Vector([1,d,f,g,g]):
> psirels:=`minus`(`union`(convert(evalm(M1s&*psi0-d*psi0),set),convert(evalm(M2s&*psi0-f*psi0),set),convert(evalm(M3s&*psi0-g*psi0),set)),{0});
```

$$\begin{aligned}psirels := \{&(2\,t-v^2+u^2)\,d+(2\,t+1)\,f+2\,(u+v)\,g-d\,f,\\
&1+(2\,t+1)\,d+(t\,v^2+2\,v^2-2\,u\,v-2\,v\,u\,t+1+2\,t+u^2\,t)\,f+2\,(v\,t+2\,v-u\,t)\,g-f^2,\\
&1+\left(\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)d+t\,f+2\,v\,g-g^2,\ 1\\
&+\left(-\frac{5}{2}v^2+u\,v+\frac{3}{2}u^2+2\,t-\frac{1}{2}v^4+v^3\,u+t\,v^2-u^3\,v-2\,v\,u\,t+\frac{1}{2}u^4+u^2\,t\right)d+(2\,t-v^2+u^2)\,f\\
&+2\left(\frac{3}{2}v+\frac{1}{2}u-v\,t+u\,t+\frac{1}{2}v^3-\frac{1}{2}v^2\,u-\frac{1}{2}u^2\,v+\frac{1}{2}u^3\right)g-d^2,\\
&(u+v)\,d+(v\,t+2\,v-u\,t)\,f+t\,g+(1+t)\,g-f\,g,\ \left(-\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)d+(1+t)\,f+u\,g+v\,g-g^2,\\
&\left(\frac{3}{2}v+\frac{1}{2}u-v\,t+u\,t+\frac{1}{2}v^3-\frac{1}{2}v^2\,u-\frac{1}{2}u^2\,v+\frac{1}{2}u^3\right)d+(u+v)\,f+\left(\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)g\\
&+\left(-\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)g-d\,g\}\end{aligned}$$

We combine the nontrivial relations obtained so far.

```
> SandNrels:=`union`(psirels,{dio1,orthrel});
```

$$\begin{aligned}SandNrels := \{&f-d+d\,f-2\,g^2,\ (2\,t-v^2+u^2)\,d+(2\,t+1)\,f+2\,(u+v)\,g-d\,f,\\
&1+(2\,t+1)\,d+(t\,v^2+2\,v^2-2\,u\,v-2\,v\,u\,t+1+2\,t+u^2\,t)\,f+2\,(v\,t+2\,v-u\,t)\,g-f^2,\\
&1+\left(\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)d+t\,f+2\,v\,g-g^2,\ 1\\
&+\left(-\frac{5}{2}v^2+u\,v+\frac{3}{2}u^2+2\,t-\frac{1}{2}v^4+v^3\,u+t\,v^2-u^3\,v-2\,v\,u\,t+\frac{1}{2}u^4+u^2\,t\right)d+(2\,t-v^2+u^2)\,f\\
&+2\left(\frac{3}{2}v+\frac{1}{2}u-v\,t+u\,t+\frac{1}{2}v^3-\frac{1}{2}v^2\,u-\frac{1}{2}u^2\,v+\frac{1}{2}u^3\right)g-d^2,\\
&(u+v)\,d+(v\,t+2\,v-u\,t)\,f+t\,g+(1+t)\,g-f\,g,\ \left(-\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)d+(1+t)\,f+u\,g+v\,g-g^2,\\
&\left(\frac{3}{2}v+\frac{1}{2}u-v\,t+u\,t+\frac{1}{2}v^3-\frac{1}{2}v^2\,u-\frac{1}{2}u^2\,v+\frac{1}{2}u^3\right)d+(u+v)\,f+\left(\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)g\end{aligned}$$

$$+\left(-\frac{1}{2}+t+\frac{u^2}{2}-\frac{v^2}{2}\right)g-d\,g,\ 2\,u^2\,t-u^2-4\,u\,v-3\,v^2-2\,t\,v^2+2\,t+1+4\,t^2\}$$

```
> SandNbasis:=Basis(SandNrels,tdeg(g,f,d,u,v,t));
```

$$SandNbasis := [u\,g-1-v\,g+f-d,\ d-d\,f-f+2\,g^2,\ 2\,u^2\,t-u^2-4\,u\,v-3\,v^2-2\,t\,v^2+2\,t+1+4\,t^2,\ f\,u\,t-f\,v\,t+f\,g-d\,u-2\,f\,v-d\,v-2\,t\,g-g,\ 8\,g\,v^2-2\,d\,u\,t+4\,f\,v\,t-2\,d\,v\,t-4\,g\,t^2-2\,f\,g-f\,u+3\,d\,u-f\,v+7\,d\,v+2\,t\,g-2\,u\,t-2\,v\,t+g+u+5\,v,\ 4\,d\,u\,v-2+4\,d\,v^2-2\,f\,d\,t+8\,g\,v\,t+4\,f\,t^2+d\,f-4\,v\,g-4\,t\,f+f-3\,d+4\,t,\ 2\,f\,d\,v-2\,g\,d\,t-d\,u\,t+d\,v\,t-2\,f\,g+d\,g-f\,u+d\,u+f\,v+d\,v+g,\ d\,u^2+2-d\,v^2-d\,f+4\,v\,g+2\,t\,f+2\,d\,t-f+2\,d,\ f\,u^2+4\,g\,f\,v-f\,v^2-2\,f\,d\,t-2\,f^2+d\,f+2\,t\,f+2\,f-d,\ 2\,f\,d\,u-2\,g\,d\,t-d\,u\,t+d\,v\,t+2\,f\,g-3\,d\,g+f\,u-d\,u-f\,v+3\,d\,v-3\,g,\ f^2\,d-4\,g\,f\,v-4\,g\,d\,v-2\,f^2\,t+f^2-2\,d^2+2\,d\,t-f-d,\ g\,f\,d-d^2\,u-d^2\,v-2\,g\,f\,t-2\,g\,d\,t-f\,g-f\,u-f\,v,\ 8\,d\,t\,v^2-2\,f\,d\,t^2+8\,g\,v\,t^2+4\,f\,t^3-8\,g\,d\,v+8\,d\,v^2+f\,d\,t+4\,d^2\,t-4\,g\,v\,t-4\,f\,t^2-4\,d\,t^2+3\,d\,f-2\,d^2-4\,v\,g-t\,f+d\,t+4\,t^2+f-d-2\,t,\ 8\,g\,f\,v\,t-4\,f\,d\,t^2-4\,g\,f\,v+4\,f\,u\,v+4\,f\,v^2-4\,f^2\,t+4\,f\,d\,t+2\,f^2-d\,f-2\,d\,t-3\,f+d,\ -4\,d\,g-9\,f\,g+16\,t\,g+16\,g\,f\,t^2+8\,g\,d\,t^2-8\,f\,v\,t^2+4\,g\,f^2+f^2\,u-7\,f^2\,v-4\,g\,f\,t-4\,g\,d\,t-8\,g\,t^2+8\,f^2\,v\,t^2-8\,g\,f\,t^3-8\,f\,u\,v^2-8\,f\,v^3-4\,g\,f^2\,t+5\,d\,u+13\,d\,v+10\,f\,v-2\,f\,u-6\,d\,v\,t-6\,d\,u\,t+16\,f\,v\,t+4\,d\,u\,t^2-4\,d\,v\,t^2]$$

The next step is to introduce the relations involving the roots of unity theta_i. The first goal is to show that th2 satisfies a nontrivial degree 3 (or less) polynomial in Q[d].

```
> psith0:=Vector([1,d*th1,f*th2,g*th3,g*th3]):
> Thetarels:=simplify(subs({h1=1/2*(f-d-1)},[th1^2*S[2,2]-(sum('M1[k
  ,2]*psith0[k]',k=1..5)),th1*th2*S[2,3]-(sum('M1[k,3]*psith0[k]',k=
  1..5)),th1*th3*S[2,4]-(sum('M1[k,4]*psith0[k]',k=1..5)),th3*th2*S[
  4,3]-(sum('M2[k,4]*psith0[k]',k=1..5)),th2^2*S[3,3]-(sum('M2[k,3]*
  psith0[k]',k=1..5)),th3^2*(S[4,4]+S[4,5])-(sum('(M3[k,4]+M3[k,5])*
  psith0[k]',k=1..5))]));
```

$$Thetarels := [-th1^2\,f-1-n1\,d\,th1-n2\,f\,th2-2\,n3\,g\,th3,\ -th1\,th2-n2\,d\,th1-n4\,f\,th2-2\,n5\,g\,th3,\ th1\,th3\,g-n3\,d\,th1-n5\,f\,th2-n6\,g\,th3-n7\,g\,th3,\ -th3\,th2\,g-n5\,d\,th1-n9\,f\,th2-n10\,g\,th3-n11\,g\,th3,\ th2^2\,d-1-n4\,d\,th1-n8\,f\,th2-2\,n9\,g\,th3,\ th3^2\,f-th3^2\,d-th3^2-1-d\,th1\,n6-d\,th1\,n7-f\,th2\,n11-f\,th2\,n10-2\,g\,th3\,n12-g\,th3\,n13-g\,th3\,n14]$$

We solve for th1 first, note that the denominator is not 0.

```
> solve(Thetarels[2],th1);
```

$$-\frac{n4\,f\,th2+2\,n5\,g\,th3}{th2+n2\,d}$$

We substitute back in to eliminate th1 and take at the numerators of the resulting rational functions, which give us our new relations.

```
> Thetarels2:=factor(numer(subs({th1=solve(Thetarels[2],th1)},Thetar
  els)));
```

$$
\begin{aligned}
\mathit{Thetarels2} := [&-n4^2 f^3 th2^2 - 4\, n4 f^2 th2\, n5\, g\, th3 - 4 f\, n5^2 g^2 th3^2 - th2^2 - 2\, th2\, n2\, d - n2^2 d^2 \\
&+ n1\, d\, n4 f\, th2^2 + n1\, d^2 n4 f\, th2\, n2 + 2\, n1\, d\, n5\, g\, th3\, th2 + 2\, n1\, d^2 n5\, g\, th3\, n2 - n2 f\, th2^3 \\
&- 2\, n2^2 f\, th2^2 d - n2^3 f\, th2\, d^2 - 2\, n3\, g\, th3\, th2^2 - 4\, n3\, g\, th3\, th2\, n2\, d - 2\, n3\, g\, th3\, n2^2 d^2, 0, \\
&-g\, th3\, n4 f\, th2 - 2 g^2 th3^2 n5 + n3\, d\, n4 f\, th2 + 2\, n3\, d\, n5\, g\, th3 - n5 f\, th2^2 - n5 f\, th2\, n2\, d \\
&- n6\, g\, th3\, th2 - n6\, g\, th3\, n2\, d - n7\, g\, th3\, th2 - n7\, g\, th3\, n2\, d, -th3\, th2^2 g - th3\, th2\, g\, n2\, d \\
&+ n5\, d\, n4 f\, th2 + 2\, n5^2 d\, g\, th3 - n9 f\, th2^2 - n9 f\, th2\, n2\, d - n10\, g\, th3\, th2 - n10\, g\, th3\, n2\, d \\
&- n11\, g\, th3\, th2 - n11\, g\, th3\, n2\, d, th2^3 d + th2^2 d^2 n2 - th2 - n2\, d + n4^2 d f\, th2 + 2\, n4\, d\, n5\, g\, th3 \\
&- n8 f\, th2^2 - n8 f\, th2\, n2\, d - 2\, n9\, g\, th3\, th2 - 2\, n9\, g\, th3\, n2\, d, th3^2 f\, th2 + th3^2 f\, n2\, d - th3^2 d\, th2 \\
&- th3^2 d^2 n2 - th3^2 th2 - th3^2 n2\, d - th2 - n2\, d + d\, n6\, n4 f\, th2 + 2\, d\, n6\, n5\, g\, th3 + d\, n7\, n4 f\, th2 \\
&+ 2\, d\, n7\, n5\, g\, th3 - f\, th2^2 n11 - f\, th2\, n11\, n2\, d - f\, th2^2 n10 - f\, th2\, n10\, n2\, d - 2\, g\, th3\, n12\, th2 \\
&- 2\, g\, th3\, n12\, n2\, d - g\, th3\, n13\, th2 - g\, th3\, n13\, n2\, d - g\, th3\, n14\, th2 - g\, th3\, n14\, n2\, d]
\end{aligned}
$$

```
> map(degree,Thetarels2,th3);
```

$$[2, -\infty, 2, 1, 1, 2]$$

Notice: either the denominator is non-zero, or th2 satisfies a degree 2 polynomial.

```
> solve(Thetarels2[4],th3);
```

$$-\frac{f\, th2\, (n9\, th2 + n9\, n2\, d - n5\, d\, n4)}{g\, (-2\, n5^2 d + th2^2 + th2\, n2\, d + n10\, th2 + n10\, n2\, d + n11\, th2 + n11\, n2\, d)}$$

```
> Thetarels3:=factor(numer(subs({th3=solve(Thetarels2[4],th3)},Theta
  rels2)));
```

$$
\begin{aligned}
\mathit{Thetarels3} := [&(th2 + n2\, d)^2 (4\, n4 f^3 th2^3 n5\, n9 - 2\, n4^2 f^3 th2^2 n10\, n11 - 2\, n2 f\, th2^3 n10\, n11 \\
&+ 4\, d\, n2 f\, th2^2 n5^2 n11 + 4\, d\, n2\, n3 f\, th2^3 n9 - 4\, d\, n2^2 f\, th2^2 n10\, n11 + 4\, n5^2 d\, th2^2 - 2\, th2^3 n2\, d \\
&- th2^2 n2^2 d^2 - 2\, n10\, th2^2 n11 - n10^2 n2^2 d^2 - n11^2 n2^2 d^2 + 4\, n5^2 d^2 th2\, n2 + 4\, n5^2 d\, n10\, th2 \\
&+ 4\, n5^2 d^2 n10\, n2 + 4\, n5^2 d\, n11\, th2 + 4\, n5^2 d^2 n11\, n2 - 4\, th2^2 n10\, n2\, d - 4\, th2^2 n11\, n2\, d \\
&- 2\, th2\, n2^2 d^2 n10 - 2\, th2\, n2^2 d^2 n11 - 2\, n10^2 th2\, n2\, d - 2\, n10\, n2^2 d^2 n11 - 2\, n11^2 th2\, n2\, d \\
&- n4^2 f^3 th2^4 + 4\, n4 f^3 th2^2 n5\, n9\, n10 + 4\, n4 f^3 th2^2 n5\, n9\, n11 - 2\, d^2 n1\, n4 f\, th2^2 n5^2 \\
&+ 4\, d^2 n1\, n5^3 f\, th2\, n9 + 4\, d^2 n3 f\, th2\, n5^3 n4 - 2\, d^2 n2^3 f\, th2\, n10\, n11 + 4\, d^2 n2^2 f\, th2\, n5^2 n10 \\
&+ 4\, d^2 n2^2 f\, th2\, n5^2 n11 + 2\, d^2 n2^2 n3 f\, th2^2 n9 + d^2 n2\, n1\, n4 f\, th2^3 + d\, n1\, n4 f\, th2^2 n10^2 \\
&+ d\, n1\, n4 f\, th2^2 n11^2 + 2\, d\, n1\, n4 f\, th2^3 n10 + 2\, d\, n1\, n4 f\, th2^3 n11 - 2\, d\, n1\, n5 f\, th2^3 n9 \\
&- 4\, d\, n3 f\, th2^2 n9\, n5^2 - 2\, d\, n3 f\, th2^3 n5\, n4 + 4\, d\, n2 f\, th2^2 n5^2 n10 - 2\, d^2 n1\, n4 f\, th2\, n5^2 n11 \\
&+ 2\, d^2 n2^2 n3 f\, th2\, n9\, n11 + 2\, d^2 n2^2 n3 f\, th2\, n9\, n10 + 2\, d^2 n2\, n1\, n4 f\, th2^2 n10 \\
&- 2\, d^2 n2\, n1\, n5 f\, th2^2 n9 + d^2 n2\, n1\, n4 f\, th2\, n11^2 - 4\, d^2 n2\, n3 f\, th2\, n9\, n5^2 \\
&+ 2\, d^2 n2\, n1\, n4 f\, th2^2 n11 - 2\, d^2 n2\, n3 f\, th2^2 n5\, n4 + d^2 n2\, n1\, n4 f\, th2\, n10^2 \\
&+ 2\, d\, n1\, n4 f\, th2^2 n10\, n11 - 2\, d\, n1\, n5 f\, th2^2 n9\, n10 - 2\, d\, n1\, n5 f\, th2^2 n9\, n11
\end{aligned}
$$

$$- 2 d n3 f th2^{2} n5 n4 n10 - 2 d n3 f th2^{2} n5 n4 n11 + 4 d n2 n3 f th2^{2} n9 n11$$

$$+ 4 d n2 n3 f th2^{2} n9 n10 + 2 n3 f th2^{3} n9 n10 + 2 n3 f th2^{3} n9 n11 - th2^{4} - 4 n10 th2 n11 n2 d$$

$$- n4^{2} f^{3} th2^{2} n10^{2} - n4^{2} f^{3} th2^{2} n11^{2} - 2 n4^{2} f^{3} th2^{3} n10 - 2 n4^{2} f^{3} th2^{3} n11 - 4 f^{3} n5^{2} th2^{2} n9^{2}$$

$$+ 2 n3 f th2^{4} n9 - 2 n2 f th2^{4} n11 - n2 f th2^{3} n10^{2} - n2 f th2^{3} n11^{2} - 2 n2 f th2^{4} n10$$

$$- 2 d^{2} n2^{3} f th2^{2} n11 - d^{2} n2^{3} f th2 n11^{2} - 2 d^{2} n2^{3} f th2^{2} n10 - d^{2} n2^{3} f th2 n10^{2}$$

$$+ 4 d^{2} n2^{2} f th2^{2} n5^{2} - 4 d^{2} n2 f th2 n5^{4} + d n1 n4 f th2^{4} - 2 d n2^{2} f th2^{2} n11^{2} - 2 d n2^{2} f th2^{2} n10^{2}$$

$$- 4 d n2^{2} f th2^{3} n10 - 4 d n2^{2} f th2^{3} n11 + 4 d n2 f th2^{3} n5^{2} - 2 d^{2} n1 n4 f th2 n5^{2} n10$$

$$- 2 d^{2} n2 n1 n5 f th2 n9 n11 - 2 d^{2} n2 n3 f th2 n5 n4 n10 - 2 d^{2} n2 n3 f th2 n5 n4 n11$$

$$- 2 d^{2} n2 n1 n5 f th2 n9 n10 + 2 d^{2} n2 n1 n4 f th2 n10 n11 - 4 n5^{4} d^{2} - 2 th2^{3} n10 - 2 th2^{3} n11$$

$$- n10^{2} th2^{2} - n11^{2} th2^{2} - 2 n2^{2} f th2^{4} d - n2 f th2^{5} - n2^{3} f th2^{3} d^{2}), 0, -f th2 (th2 + n2 d) ($$

$$2 n5 th2^{2} n10 n11 + d^{2} n2^{2} n5 n10^{2} + d^{2} n2^{2} n5 n11^{2} - 4 d^{2} n2 n5^{3} n11 - 4 d^{2} n2 n5^{3} n10$$

$$- 4 d n5^{3} th2 n11 - 4 d n5^{3} th2 n10 - d n2 n4 f th2 n9 n10 - d n2 n4 f th2 n9 n11$$

$$+ d^{2} n2 n6 n5 n4 n11 - 2 d^{2} n2 n3 n4 th2 n10 - 2 d^{2} n2 n3 n4 th2 n11 + 2 d^{2} n2 n3 n5 n9 n11$$

$$+ d^{2} n2 n6 n5 n4 n10 + 2 d^{2} n2 n3 n5 n9 th2 + d^{2} n2 n7 n5 n4 th2 + d^{2} n2 n6 n5 n4 th2$$

$$+ d^{2} n2 n7 n5 n4 n11 + 2 d^{2} n2 n3 n5 n9 n10 - 2 d^{2} n2 n3 n4 n10 n11 + d^{2} n2 n7 n5 n4 n10$$

$$- 2 d n2 n7 th2 n9 n10 - 2 d n2 n7 th2 n9 n11 + 2 d n2 n5 f th2 n9^{2} - 2 d n2 n6 th2 n9 n10$$

$$- 2 d n2 n6 th2 n9 n11 + d n7 th2 n5 n4 n11 - 2 d n4 f th2 n9 n5^{2} + d n6 th2 n5 n4 n10$$

$$+ d n6 th2 n5 n4 n11 + d n7 th2 n5 n4 n10 - d n2 n4 f th2^{2} n9 + d n4^{2} f th2 n5 n10$$

$$+ d n4^{2} f th2 n5 n11 - 2 d n3 n4 n10 th2 n11 + 2 d n3 n5 n9 th2 n10 + 2 d n3 n5 n9 th2 n11$$

$$+ n5 th2^{4} - 4 d^{2} n5^{3} th2 n2 + d^{2} n5 th2^{2} n2^{2} + 2 d n5 th2^{3} n2 + n5 th2^{2} n11^{2} + n5 th2^{2} n10^{2}$$

$$- n7 th2^{3} n9 - n6 th2^{3} n9 + 2 n5 th2^{3} n10 + 4 d^{2} n5^{5} + 2 d^{2} n5 th2 n2^{2} n11 + 4 d n5 th2^{2} n10 n2$$

$$+ 4 d n5 th2^{2} n11 n2 + 2 d^{2} n5 th2 n2^{2} n10 + 4 d n2 n5 th2 n10 n11 + 2 d^{2} n2^{2} n5 n10 n11$$

$$+ 2 d n2 n5 th2 n11^{2} + 2 d n2 n5 th2 n10^{2} - n4 f th2^{3} n9 + 2 n5 f th2^{2} n9^{2} - n6 th2^{2} n9 n10$$

$$- n6 th2^{2} n9 n11 - n7 th2^{2} n9 n10 - n7 th2^{2} n9 n11 - 2 d^{2} n6 n5^{3} n4 - 2 d^{2} n7 n5^{3} n4$$

$$- 4 d^{2} n3 n5^{3} n9 - d n3 n4 th2^{3} - 4 d n5^{3} th2^{2} + 2 n5 th2^{3} n11 - n4 f th2^{2} n9 n10$$

$$- n4 f th2^{2} n9 n11 + 2 d^{2} n2 n5^{2} n6 n9 + 2 d^{2} n2 n5^{2} n7 n9 - d^{2} n2^{2} n10 n7 n9$$

$$+ 2 d^{2} n3 n4 n5^{2} n10 + 2 d^{2} n3 n4 n5^{2} n11 - d^{2} n2^{2} n11 n7 n9 - d^{2} n2^{2} n6 n9 n10$$

$$- d^{2} n2^{2} n7 n9 th2 - d^{2} n2^{2} n6 n9 th2 - d^{2} n2^{2} n6 n9 n11 - d^{2} n2 n3 n4 n11^{2} - d^{2} n2 n3 n4 th2^{2}$$

$$- d^{2} n2 n3 n4 n10^{2} + 2 d^{2} n3 n4 n5^{2} th2 + d n4^{2} f th2^{2} n5 - 2 d n3 n4 th2^{2} n10$$

$$- 2 d n3 n4 th2^{2} n11 - d n3 n4 n10^{2} th2 - d n3 n4 n11^{2} th2 + 2 d n3 n5 n9 th2^{2}$$

$$+ 2 d n6 th2 n9 n5^{2} + d n6 th2^{2} n5 n4 + 2 d n7 th2 n9 n5^{2} + d n7 th2^{2} n5 n4 - 2 d n2 n7 th2^{2} n9$$

$$- 2 d n2 n6 th2^{2} n9), 0, (th2 + n2 d) (2 n5^{2} d - n11 th2 - n10 n2 d - n11 n2 d - th2 n2 d$$

$$+ th2^{3} d^{2} n2 - d\, n2\, n8\, f\, th2\, n10 - d\, n2\, n8\, f\, th2\, n11 - 4\, d\, n5\, n4\, f\, th2\, n9 + 2\, d\, n2\, n9^{2} f\, th2$$
$$- d\, n2\, n8\, f\, th2^{2} + 2\, d\, n8\, f\, th2\, n5^{2} + d\, n4^{2} f\, th2\, n10 + d\, n4^{2} f\, th2\, n11 + th2^{4} d - n8\, f\, th2^{2} n10$$
$$- n8\, f\, th2^{2} n11 + d^{2} th2^{2} n10\, n2 + d^{2} th2^{2} n11\, n2 + d\, n4^{2} f\, th2^{2} - 2\, d^{2} n5^{2} th2^{2} - th2^{2} - n8\, f\, th2^{3}$$
$$+ 2\, n9^{2} f\, th2^{2} + d\, th2^{3} n11 + d\, th2^{3} n10 - n10\, th2), -(th2 + n2\, d)\, (2\, th2^{2} n11\, g^{2} n10$$
$$+ 2\, th2^{3} n11\, g^{2} + th2^{2} n11^{2} g^{2} + 2\, th2^{3} n10\, g^{2} + th2^{2} n10^{2} g^{2} - 2\, d^{2} n2^{2} f\, th2^{2} n12\, g^{2} n9$$
$$- d^{2} n2^{2} f\, th2^{2} n14\, g^{2} n9 + 3\, d^{2} n2^{2} f\, th2\, n11\, g^{2} n10^{2} + 4\, d^{2} n2^{2} f\, th2^{2} n11\, g^{2} n10$$
$$+ 3\, d^{2} n2^{2} f\, th2\, n11^{2} g^{2} n10 + 2\, d^{2} n2\, f^{3} th2^{2} n9\, n5\, n4 - 4\, d^{2} n2\, f\, th2^{2} n11\, g^{2} n5^{2}$$
$$- 4\, d^{2} n2\, f\, th2\, n11^{2} g^{2} n5^{2} - 4\, d^{2} n2\, f\, th2^{2} n10\, g^{2} n5^{2} - 4\, d^{2} n2\, f\, th2\, n10^{2} g^{2} n5^{2}$$
$$- d^{2} n2\, n6\, n4\, f\, th2^{3} g^{2} - d^{2} n2\, n7\, n4\, f\, th2^{3} g^{2} + d\, f\, th2^{3} n14\, g^{2} n5\, n4 - 2\, d\, n6\, n4\, f\, th2^{3} g^{2} n10$$
$$- 2\, d\, n6\, n4\, f\, th2^{3} g^{2} n11 - d\, n6\, n4\, f\, th2^{2} g^{2} n10^{2} - d\, n6\, n4\, f\, th2^{2} g^{2} n11^{2} + 2\, d\, n6\, n5\, f\, th2^{3} g^{2} n9$$
$$- 4\, d\, n2\, f\, th2^{3} n12\, g^{2} n9 - 2\, d\, n2\, f\, th2^{3} n13\, g^{2} n9 - 2\, d\, n2\, f\, th2^{3} n14\, g^{2} n9$$
$$+ 6\, d\, n2\, f\, th2^{2} n11\, g^{2} n10^{2} + 2\, d^{2} n7\, n4\, f\, th2\, g^{2} n5^{2} n10 + 2\, d^{2} n7\, n4\, f\, th2\, g^{2} n5^{2} n11$$
$$+ 2\, d^{2} n6\, n4\, f\, th2\, g^{2} n5^{2} n10 + 2\, d^{2} n6\, n4\, f\, th2\, g^{2} n5^{2} n11 - 2\, d^{2} n2^{2} f\, th2\, n12\, g^{2} n9\, n10$$
$$- 2\, d^{2} n2^{2} f\, th2\, n12\, g^{2} n9\, n11 - d^{2} n2^{2} f\, th2\, n13\, g^{2} n9\, n10 - d^{2} n2^{2} f\, th2\, n13\, g^{2} n9\, n11$$
$$- d^{2} n2^{2} f\, th2\, n14\, g^{2} n9\, n10 - d^{2} n2^{2} f\, th2\, n14\, g^{2} n9\, n11 - 8\, d^{2} n2\, f\, th2\, n11\, g^{2} n5^{2} n10$$
$$+ 2\, d^{2} n2\, n6\, n5\, f\, th2^{2} g^{2} n9 - d^{2} n2\, n7\, n4\, f\, th2\, g^{2} n10^{2} - d^{2} n2\, n7\, n4\, f\, th2\, g^{2} n11^{2}$$
$$- 2\, d^{2} n2\, n7\, n4\, f\, th2^{2} g^{2} n10 - 2\, d^{2} n2\, n7\, n4\, f\, th2^{2} g^{2} n11 + 2\, d^{2} n2\, n7\, n5\, f\, th2^{2} g^{2} n9$$
$$+ 4\, d^{2} n2\, f\, th2\, n12\, g^{2} n9\, n5^{2} + 2\, d^{2} n2\, f\, th2^{2} n12\, g^{2} n5\, n4 + 2\, d^{2} n2\, f\, th2\, n13\, g^{2} n9\, n5^{2}$$
$$+ d^{2} n2\, f\, th2^{2} n13\, g^{2} n5\, n4 + 2\, d^{2} n2\, f\, th2\, n14\, g^{2} n9\, n5^{2} + d^{2} n2\, f\, th2^{2} n14\, g^{2} n5\, n4$$
$$- d^{2} n2\, n6\, n4\, f\, th2\, g^{2} n11^{2} - d^{2} n2\, n6\, n4\, f\, th2\, g^{2} n10^{2} - 2\, d^{2} n2\, n6\, n4\, f\, th2^{2} g^{2} n10$$
$$- 2\, d^{2} n2\, n6\, n4\, f\, th2^{2} g^{2} n11 - 2\, d^{2} n2\, n6\, n4\, f\, th2\, g^{2} n10\, n11 + 2\, d^{2} n2\, n6\, n5\, f\, th2\, g^{2} n9\, n10$$
$$+ 2\, d^{2} n2\, n6\, n5\, f\, th2\, g^{2} n9\, n11 - 2\, d^{2} n2\, n7\, n4\, f\, th2\, g^{2} n10\, n11 + 2\, d^{2} n2\, n7\, n5\, f\, th2\, g^{2} n9\, n10$$
$$+ 2\, d^{2} n2\, n7\, n5\, f\, th2\, g^{2} n9\, n11 + 2\, d^{2} n2\, f\, th2\, n12\, g^{2} n5\, n4\, n10 + 2\, d^{2} n2\, f\, th2\, n12\, g^{2} n5\, n4\, n11$$
$$+ d^{2} n2\, f\, th2\, n13\, g^{2} n5\, n4\, n10 + d^{2} n2\, f\, th2\, n13\, g^{2} n5\, n4\, n11 + d^{2} n2\, f\, th2\, n14\, g^{2} n5\, n4\, n10$$
$$+ d^{2} n2\, f\, th2\, n14\, g^{2} n5\, n4\, n11 - 2\, d\, n6\, n4\, f\, th2^{2} g^{2} n10\, n11 + 2\, d\, n6\, n5\, f\, th2^{2} g^{2} n9\, n10$$
$$+ 2\, d\, n6\, n5\, f\, th2^{2} g^{2} n9\, n11 + 2\, d\, n7\, n5\, f\, th2^{2} g^{2} n9\, n10 + 2\, d\, n7\, n5\, f\, th2^{2} g^{2} n9\, n11$$
$$- 2\, d\, n7\, n4\, f\, th2^{2} g^{2} n10\, n11 + 2\, d\, f\, th2^{2} n12\, g^{2} n5\, n4\, n10 + 2\, d\, f\, th2^{2} n12\, g^{2} n5\, n4\, n11$$
$$+ d\, f\, th2^{2} n13\, g^{2} n5\, n4\, n10 + d\, f\, th2^{2} n13\, g^{2} n5\, n4\, n11 + d\, f\, th2^{2} n14\, g^{2} n5\, n4\, n10$$
$$+ d\, f\, th2^{2} n14\, g^{2} n5\, n4\, n11 - 2\, d\, n2\, f\, th2^{2} n14\, g^{2} n9\, n10 - 2\, d\, n2\, f\, th2^{2} n14\, g^{2} n9\, n11$$
$$- 4\, d\, n2\, f\, th2^{2} n12\, g^{2} n9\, n10 - 4\, d\, n2\, f\, th2^{2} n12\, g^{2} n9\, n11 - 2\, d\, n2\, f\, th2^{2} n13\, g^{2} n9\, n10$$
$$- 2\, d\, n2\, f\, th2^{2} n13\, g^{2} n9\, n11 - 2\, d^{3} f^{2} th2^{2} n9\, n5\, n4\, n2 - d^{2} n2^{2} f\, th2^{2} n13\, g^{2} n9$$
$$- 2\, d^{2} f^{2} th2^{2} n9\, n5\, n4\, n2 - 4\, d^{2} n6\, n5^{3} f\, th2\, g^{2} n9 + 2\, d^{2} n7\, n4\, f\, th2^{2} g^{2} n5^{2}$$

$$-4\,d^2\,n7\,n5^3\,f\,th2\,g^2\,n9-4\,d^2\,f\,th2\,n12\,g^2\,n5^3\,n4-2\,d^2\,f\,th2\,n13\,g^2\,n5^3\,n4$$
$$-2\,d^2\,f\,th2\,n14\,g^2\,n5^3\,n4+2\,d^2\,n6\,n4\,f\,th2^2\,g^2\,n5^2-2\,d^2\,f^2\,th2^3\,n9\,n5\,n4+f^2\,th2^4\,n9^2$$
$$-f^3\,th2^4\,n9^2+4\,d^2\,n5^4\,g^2-2\,f\,th2^4\,n12\,g^2\,n9-f\,th2^4\,n13\,g^2\,n9-f\,th2^4\,n14\,g^2\,n9$$
$$+4\,f\,th2^4\,n11\,g^2\,n10+3\,f\,th2^3\,n11\,g^2\,n10^2+3\,f\,th2^3\,n11^2\,g^2\,n10+d^3\,f^2\,th2^2\,n9^2\,n2^2$$
$$+d^3\,f^2\,th2^2\,n5^2\,n4^2-d^2\,n2^2\,f^3\,th2^2\,n9^2+2\,d^2\,n2^2\,n11\,g^2\,n10+2\,d^2\,n2^2\,g^2\,th2\,n10$$
$$+d^2\,f^2\,th2^2\,n5^2\,n4^2+d^2\,f^2\,th2^2\,n9^2\,n2^2-d^2\,f^3\,th2^2\,n5^2\,n4^2-4\,d^2\,n2\,g^2\,n5^2\,n10$$
$$-4\,d^2\,n2\,g^2\,n5^2\,n11+2\,d^2\,n2\,f^2\,th2^3\,n9^2-4\,d^2\,n2\,g^2\,n5^2\,th2-4\,d\,th2\,g^2\,n5^2\,n10$$
$$-4\,d\,th2\,g^2\,n5^2\,n11+4\,d\,n2\,th2^2\,n10\,g^2+4\,d\,n2\,th2^2\,n11\,g^2+2\,d\,n2\,th2\,n10^2\,g^2$$
$$+2\,d\,n2\,th2\,n11^2\,g^2-2\,d\,n2\,f^3\,th2^3\,n9^2+2\,d\,n2\,f^2\,th2^3\,n9^2+8\,d\,n2\,f\,th2^3\,n11\,g^2\,n10$$
$$+6\,d\,n2\,f\,th2^2\,n11^2\,g^2\,n10-8\,d\,f\,th2^2\,n11\,g^2\,n5^2\,n10-2\,d\,n7\,n4\,f\,th2^3\,g^2\,n10$$
$$-2\,d\,n7\,n4\,f\,th2^3\,g^2\,n11-d\,n7\,n4\,f\,th2^2\,g^2\,n10^2-d\,n7\,n4\,f\,th2^2\,g^2\,n11^2+2\,d\,n7\,n5\,f\,th2^3\,g^2\,n9$$
$$+4\,d\,f\,th2^2\,n12\,g^2\,n9\,n5^2+2\,d\,f\,th2^3\,n12\,g^2\,n5\,n4+2\,d\,f\,th2^2\,n13\,g^2\,n9\,n5^2$$
$$+d\,f\,th2^3\,n13\,g^2\,n5\,n4+2\,d\,f\,th2^2\,n14\,g^2\,n9\,n5^2-2\,f\,th2^3\,n12\,g^2\,n9\,n10$$
$$-2\,f\,th2^3\,n12\,g^2\,n9\,n11-f\,th2^3\,n13\,g^2\,n9\,n10-f\,th2^3\,n13\,g^2\,n9\,n11-f\,th2^3\,n14\,g^2\,n9\,n10$$
$$-f\,th2^3\,n14\,g^2\,n9\,n11+d^2\,n2^2\,f\,th2\,n10^3\,g^2+4\,d^2\,f\,th2\,n11\,g^2\,n5^4+4\,d^2\,f\,th2\,n10\,g^2\,n5^4$$
$$+2\,d^2\,n2^2\,f\,th2^2\,n10^2\,g^2+d^2\,n2^2\,f\,th2\,n11^3\,g^2+2\,d^2\,n2^2\,f\,th2^2\,n11^2\,g^2+d^2\,n2^2\,f\,g^2\,th2^3\,n11$$
$$+d^2\,n2^2\,f\,g^2\,th2^3\,n10+2\,d\,n4\,f^3\,th2^3\,n5\,n9-2\,d\,f^2\,th2^3\,n9\,n5\,n4-4\,d\,f\,th2^3\,n11\,g^2\,n5^2$$
$$-4\,d\,f\,th2^2\,n11^2\,g^2\,n5^2-4\,d\,f\,th2^3\,n10\,g^2\,n5^2-4\,d\,f\,th2^2\,n10^2\,g^2\,n5^2-d\,n6\,n4\,f\,th2^4\,g^2$$
$$-d\,n7\,n4\,f\,th2^4\,g^2+4\,d\,n2\,th2\,n11\,g^2\,n10+2\,d\,n2\,f\,th2^4\,n11\,g^2+2\,d\,n2\,f\,th2^2\,n11^3\,g^2$$
$$+4\,d\,n2\,f\,th2^3\,n11^2\,g^2+2\,d\,n2\,f\,th2^4\,n10\,g^2+2\,d\,n2\,f\,th2^2\,n10^3\,g^2+4\,d\,n2\,f\,th2^3\,n10^2\,g^2$$
$$+2\,d^2\,n2^2\,g^2\,th2\,n11+f\,th2^5\,n11\,g^2+2\,f\,th2^4\,n11^2\,g^2+f\,th2^5\,n10\,g^2+2\,f\,th2^4\,n10^2\,g^2$$
$$+f\,th2^3\,n10^3\,g^2+f\,th2^3\,n11^3\,g^2+d^2\,n2^2\,n11^2\,g^2+d^2\,n2^2\,n10^2\,g^2+d^2\,n2^2\,g^2\,th2^2$$
$$+d\,f^2\,th2^4\,n9^2-4\,d\,th2^2\,g^2\,n5^2+2\,d\,n2\,th2^3\,g^2+th2^4\,g^2)]$$

We remove spurious factors like (th2+n2d)

```
> Thetarels3red:=collect(simplify([Thetarels3[1]/(th2+n2*d)^2,Thetar
  els3[3]/(f*th2*(th2+n2*d)),Thetarels3[5]/(th2+n2*d),Thetarels3[6]/
  (th2+n2*d)]),th2);
```

$$Thetarels3red := [-n2\,f\,th2^5$$
$$+(-1+d\,n1\,n4\,f-f^3\,n4^2+2\,n3\,f\,n9-2\,n2\,f\,n11-2\,n2^2\,f\,d-2\,n2\,f\,n10)\,th2^4+(4\,n4\,f^3\,n5\,n9$$
$$-2\,n4^2\,f^3\,n11+2\,n3\,f\,n9\,n11+4\,d\,n2\,n3\,f\,n9-n2\,f\,n10^2-n2\,f\,n11^2+2\,n3\,f\,n9\,n10$$
$$+d^2\,n2\,n1\,n4\,f-2\,n4^2\,f^3\,n10-2\,n10+2\,d\,n1\,n4\,f\,n10-2\,n2\,f\,n10\,n11-4\,d\,n2^2\,f\,n10$$
$$-4\,d\,n2^2\,f\,n11-2\,d\,n3\,f\,n5\,n4-n2^3\,f\,d^2-2\,n2\,d-2\,d\,n1\,n5\,f\,n9+4\,d\,n2\,f\,n5^2-2\,n11$$
$$+2\,d\,n1\,n4\,f\,n11)\,th2^3+(-n2^2\,d^2+4\,n5^2\,d-4\,n10\,n2\,d-4\,n11\,n2\,d-n10^2-n11^2-2\,n11\,n10$$
$$-n4^2\,f^3\,n10^2-n4^2\,f^3\,n11^2-4\,f^3\,n5^2\,n9^2+2\,d^2\,n2\,n1\,n4\,f\,n10-2\,d^2\,n2\,n1\,n5\,f\,n9$$

$$+2\, d^2\, n2\, n1\, n4\, f\, n11 - 2\, d^2\, n2\, n3\, f\, n5\, n4 + 2\, d\, n1\, n4\, f\, n10\, n11 - 2\, d\, n1\, n5\, f\, n9\, n10$$
$$-2\, d\, n1\, n5\, f\, n9\, n11 - 2\, d\, n3\, f\, n5\, n4\, n10 - 2\, d\, n3\, f\, n5\, n4\, n11 + 4\, d\, n2\, n3\, f\, n9\, n11$$
$$+4\, d\, n2\, n3\, f\, n9\, n10 - 2\, n4^2\, f^3\, n10\, n11 - 2\, d^2\, n2^3\, f\, n11 - 2\, d^2\, n2^3\, f\, n10 + 4\, d^2\, n2^2\, f\, n5^2$$
$$-2\, d\, n2^2\, f\, n11^2 - 2\, d\, n2^2\, f\, n10^2 + 4\, d\, n2\, f\, n5^2\, n11 - 4\, d\, n2^2\, f\, n10\, n11 + 4\, n4\, f^3\, n5\, n9\, n10$$
$$+4\, n4\, f^3\, n5\, n9\, n11 - 2\, d^2\, n1\, n4\, f\, n5^2 + 2\, d^2\, n2^2\, n3\, f\, n9 + d\, n1\, n4\, f\, n10^2 + d\, n1\, n4\, f\, n11^2$$
$$-4\, d\, n3\, f\, n9\, n5^2 + 4\, d\, n2\, f\, n5^2\, n10)\, th2^2 + (4\, n5^2\, d\, n10 + 4\, n5^2\, d\, n11 - 2\, n10^2\, n2\, d$$
$$-2\, n11^2\, n2\, d + 4\, n2\, d^2\, n5^2 - 2\, n2^2\, d^2\, n10 - 2\, n2^2\, d^2\, n11 - 2\, d^2\, n1\, n4\, f\, n5^2\, n11$$
$$+2\, d^2\, n2^2\, n3\, f\, n9\, n11 + 2\, d^2\, n2^2\, n3\, f\, n9\, n10 + d^2\, n2\, n1\, n4\, f\, n11^2 - 4\, d^2\, n2\, n3\, f\, n9\, n5^2$$
$$+d^2\, n2\, n1\, n4\, f\, n10^2 - 2\, d^2\, n1\, n4\, f\, n5^2\, n10 - 2\, d^2\, n2\, n1\, n5\, f\, n9\, n11 - 2\, d^2\, n2\, n3\, f\, n5\, n4\, n10$$
$$-2\, d^2\, n2\, n3\, f\, n5\, n4\, n11 - 2\, d^2\, n2\, n1\, n5\, f\, n9\, n10 + 2\, d^2\, n2\, n1\, n4\, f\, n10\, n11 - 4\, n10\, n11\, n2\, d$$
$$-d^2\, n2^3\, f\, n11^2 - d^2\, n2^3\, f\, n10^2 - 4\, d^2\, n2\, f\, n5^4 + 4\, d^2\, n1\, n5^3\, f\, n9 + 4\, d^2\, n3\, f\, n5^3\, n4$$
$$-2\, d^2\, n2^3\, f\, n10\, n11 + 4\, d^2\, n2^2\, f\, n5^2\, n10 + 4\, d^2\, n2^2\, f\, n5^2\, n11)\, th2 - 4\, n5^4\, d^2 + 4\, n5^2\, d^2\, n10\, n2$$
$$-n10^2\, n2^2\, d^2 + 4\, n5^2\, d^2\, n11\, n2 - 2\, n10\, n2^2\, d^2\, n11 - n11^2\, n2^2\, d^2, -d^2\, n2^2\, n5\, n10^2$$
$$-d^2\, n2^2\, n5\, n11^2 + 4\, d^2\, n2\, n5^3\, n11 + 4\, d^2\, n2\, n5^3\, n10 - d^2\, n2\, n6\, n5\, n4\, n11$$
$$-2\, d^2\, n2\, n3\, n5\, n9\, n11 - d^2\, n2\, n6\, n5\, n4\, n10 - d^2\, n2\, n7\, n5\, n4\, n11 - 2\, d^2\, n2\, n3\, n5\, n9\, n10$$
$$+2\, d^2\, n2\, n3\, n4\, n10\, n11 - d^2\, n2\, n7\, n5\, n4\, n10 - n5\, th2^4 - 4\, d^2\, n5^5 - 2\, d^2\, n2^2\, n5\, n10\, n11$$
$$+2\, d^2\, n6\, n5^3\, n4 + 2\, d^2\, n7\, n5^3\, n4 + 4\, d^2\, n3\, n5^3\, n9$$
$$+(n3\, d\, n4 + n6\, n9 - 2\, n5\, n10 + n4\, f\, n9 + n7\, n9 - 2\, n5\, n11 - 2\, n5\, n2\, d)\, th2^3 + (2\, d\, n3\, n4\, n11$$
$$-d\, n4^2\, f\, n5 + d^2\, n2\, n3\, n4 + n6\, n9\, n11 + n4\, f\, n9\, n11 - d\, n6\, n5\, n4 + 2\, d\, n3\, n4\, n10 - n5\, n2^2\, d^2$$
$$+n4\, f\, n9\, n10 - n5\, n11^2 - 2\, n5\, f\, n9^2 + n6\, n9\, n10 + d\, n2\, n4\, f\, n9 + n10\, n7\, n9 + 2\, d\, n2\, n6\, n9$$
$$-2\, n5\, n10\, n11 - d\, n7\, n5\, n4 + 2\, d\, n2\, n7\, n9 - 4\, d\, n5\, n11\, n2 - n5\, n10^2 - 4\, d\, n5\, n10\, n2$$
$$-2\, d\, n3\, n5\, n9 + n11\, n7\, n9 + 4\, d\, n5^3)\, th2^2 + (d\, n2\, n4\, f\, n9\, n10 + d\, n2\, n4\, f\, n9\, n11 + 4\, d\, n5^3\, n11$$
$$+4\, d\, n5^3\, n10 - 2\, d^2\, n5\, n2^2\, n10 - 2\, d^2\, n5\, n2^2\, n11 - 2\, d\, n2\, n5\, n11^2 - 2\, d\, n2\, n5\, n10^2$$
$$+d^2\, n2^2\, n7\, n9 + d^2\, n2^2\, n6\, n9 - 2\, d^2\, n3\, n4\, n5^2 + d\, n3\, n4\, n10^2 + d\, n3\, n4\, n11^2 - 2\, d\, n6\, n9\, n5^2$$
$$-2\, d\, n7\, n9\, n5^2 + 2\, d^2\, n2\, n3\, n4\, n10 + 2\, d^2\, n2\, n3\, n4\, n11 - 2\, d^2\, n2\, n3\, n5\, n9 - d^2\, n2\, n7\, n5\, n4$$
$$-d^2\, n2\, n6\, n5\, n4 + 2\, d\, n2\, n7\, n9\, n10 + 2\, d\, n2\, n7\, n9\, n11 - 2\, d\, n2\, n5\, f\, n9^2 + 2\, d\, n2\, n6\, n9\, n10$$
$$+2\, d\, n2\, n6\, n9\, n11 - d\, n7\, n5\, n4\, n11 + 2\, d\, n4\, f\, n9\, n5^2 - d\, n6\, n5\, n4\, n10 - d\, n6\, n5\, n4\, n11$$
$$-d\, n7\, n5\, n4\, n10 - d\, n4^2\, f\, n5\, n10 - d\, n4^2\, f\, n5\, n11 + 2\, d\, n3\, n4\, n10\, n11 - 2\, d\, n3\, n5\, n9\, n10$$
$$-2\, d\, n3\, n5\, n9\, n11 - 4\, d\, n2\, n5\, n10\, n11 + 4\, d^2\, n5^3\, n2)\, th2 - 2\, d^2\, n2\, n5^2\, n6\, n9$$
$$-2\, d^2\, n2\, n5^2\, n7\, n9 + d^2\, n2^2\, n10\, n7\, n9 - 2\, d^2\, n3\, n4\, n5^2\, n10 - 2\, d^2\, n3\, n4\, n5^2\, n11$$
$$+d^2\, n2^2\, n11\, n7\, n9 + d^2\, n2^2\, n6\, n9\, n10 + d^2\, n2^2\, n6\, n9\, n11 + d^2\, n2\, n3\, n4\, n11^2 + d^2\, n2\, n3\, n4\, n10^2,$$
$$th2^4\, d + (d^2\, n2 - n8\, f + n11\, d + n10\, d)\, th2^3$$

$$+ (d^2 n11\, n2 - d\, n2\, n8\, f - n8\, f\, n10 - n8\, f\, n11 + d^2 n10\, n2 + d\, n4^2 f - 2\, d^2 n5^2 - 1 + 2\, n9^2 f)\, th2^2$$
$$+ (-n11 - n2\, d + 2\, d\, n2\, n9^2 f - d\, n2\, n8\, f\, n10 - d\, n2\, n8\, f\, n11 - 4\, d\, n5\, n4\, f\, n9 + d\, n4^2 f\, n11 - n10$$
$$+ 2\, d\, n8\, f\, n5^2 + d\, n4^2 f\, n10)\, th2 + 2\, n5^2 d - n10\, n2\, d - n11\, n2\, d,\ (-f\, n10\, g^2 - f\, n11\, g^2)\, th2^5 + ($$
$$-4\, f\, n11\, g^2 n10 - 2\, d\, n2\, f\, n10\, g^2 + d\, n6\, n4\, f\, g^2 + f^3 n9^2 + d\, n7\, n4\, f\, g^2 + f\, n13\, g^2 n9 + f\, n14\, g^2 n9$$
$$- f^2 n9^2 - 2\, f\, n11^2 g^2 - 2\, d\, n2\, f\, n11\, g^2 - 2\, f\, n10^2 g^2 - g^2 - d\, f^2 n9^2 + 2\, f\, n12\, g^2 n9)\, th2^4 + ($$
$$d^2 n2\, n6\, n4\, f\, g^2 + d^2 n2\, n7\, n4\, f\, g^2 - 3\, f\, n11\, g^2 n10^2 - 3\, f\, n11^2 g^2 n10 - 2\, d^2 n2\, f^2 n9^2$$
$$+ 2\, d\, n2\, f^3 n9^2 - 2\, d\, n2\, f^2 n9^2 + 2\, d^2 f^2 n9\, n5\, n4 - f\, n10^3 g^2 - f\, n11^3 g^2 + 2\, f\, n12\, g^2 n9\, n10$$
$$+ 2\, f\, n12\, g^2 n9\, n11 + f\, n13\, g^2 n9\, n10 + f\, n13\, g^2 n9\, n11 + f\, n14\, g^2 n9\, n10 + f\, n14\, g^2 n9\, n11$$
$$- d^2 n2^2 f\, g^2 n11 - d^2 n2^2 f\, g^2 n10 - 2\, d\, n4\, f^3 n5\, n9 + 2\, d\, f^2 n9\, n5\, n4 + 4\, d\, f\, n11\, g^2 n5^2$$
$$+ 4\, d\, f\, n10\, g^2 n5^2 - 4\, d\, n2\, f\, n11^2 g^2 - 4\, d\, n2\, f\, n10^2 g^2 - 2\, n11\, g^2 - 2\, n10\, g^2$$
$$- 8\, d\, n2\, f\, n11\, g^2 n10 + 2\, d\, n7\, n4\, f\, g^2 n10 + 2\, d\, n7\, n4\, f\, g^2 n11 - 2\, d\, n7\, n5\, f\, g^2 n9$$
$$- 2\, d\, f\, n12\, g^2 n5\, n4 - d\, f\, n13\, g^2 n5\, n4 - 2\, n2\, d\, g^2 + 2\, d\, n2\, f\, n14\, g^2 n9 + 4\, d\, n2\, f\, n12\, g^2 n9$$
$$+ 2\, d\, n2\, f\, n13\, g^2 n9 - d\, f\, n14\, g^2 n5\, n4 + 2\, d\, n6\, n4\, f\, g^2 n10 + 2\, d\, n6\, n4\, f\, g^2 n11$$
$$- 2\, d\, n6\, n5\, f\, g^2 n9)\, th2^3 + (2\, d^2 n2^2 f\, n12\, g^2 n9 + d^2 n2^2 f\, n14\, g^2 n9 - 4\, d^2 n2^2 f\, n11\, g^2 n10$$
$$- 2\, d^2 n2\, f^3 n9\, n5\, n4 + 4\, d^2 n2\, f\, n11\, g^2 n5^2 + 4\, d^2 n2\, f\, n10\, g^2 n5^2 - d^3 f^2 n9^2 n2^2 - d^3 f^2 n5^2 n4^2$$
$$+ d^2 n2^2 f^3 n9^2 - d^2 f^2 n5^2 n4^2 - d^2 f^2 n9^2 n2^2 + d^2 f^3 n5^2 n4^2 - 4\, d\, n2\, n10\, g^2 - 4\, d\, n2\, n11\, g^2$$
$$- d^2 n2^2 g^2 + 4\, d\, g^2 n5^2 - 2\, d^2 n2^2 f\, n10^2 g^2 - 2\, d^2 n2^2 f\, n11^2 g^2 + 4\, d\, f\, n11^2 g^2 n5^2$$
$$+ 4\, d\, f\, n10^2 g^2 n5^2 - 2\, d\, n2\, f\, n11^3 g^2 - 2\, d\, n2\, f\, n10^3 g^2 - 2\, n11\, g^2 n10 + 2\, d^3 f^2 n9\, n5\, n4\, n2$$
$$+ d^2 n2^2 f\, n13\, g^2 n9 + 2\, d^2 f^2 n9\, n5\, n4\, n2 - 2\, d^2 n7\, n4\, f\, g^2 n5^2 - 2\, d^2 n6\, n4\, f\, g^2 n5^2$$
$$- 6\, d\, n2\, f\, n11^2 g^2 n10 + 8\, d\, f\, n11\, g^2 n5^2 n10 + d\, n7\, n4\, f\, g^2 n10^2 + d\, n7\, n4\, f\, g^2 n11^2$$
$$- 4\, d\, f\, n12\, g^2 n9\, n5^2 - 2\, d\, f\, n13\, g^2 n9\, n5^2 - 2\, d\, f\, n14\, g^2 n9\, n5^2 - 2\, d^2 n2\, n6\, n5\, f\, g^2 n9$$
$$+ 2\, d^2 n2\, n7\, n4\, f\, g^2 n10 + 2\, d^2 n2\, n7\, n4\, f\, g^2 n11 - 2\, d^2 n2\, n7\, n5\, f\, g^2 n9 - 2\, d^2 n2\, f\, n12\, g^2 n5\, n4$$
$$- d^2 n2\, f\, n13\, g^2 n5\, n4 - d^2 n2\, f\, n14\, g^2 n5\, n4 + 2\, d^2 n2\, n6\, n4\, f\, g^2 n10 + 2\, d^2 n2\, n6\, n4\, f\, g^2 n11$$
$$+ 2\, d\, n6\, n4\, f\, g^2 n10\, n11 - 2\, d\, n6\, n5\, f\, g^2 n9\, n10 - 2\, d\, n6\, n5\, f\, g^2 n9\, n11 - 2\, d\, n7\, n5\, f\, g^2 n9\, n10$$
$$- 2\, d\, n7\, n5\, f\, g^2 n9\, n11 + 2\, d\, n7\, n4\, f\, g^2 n10\, n11 - 2\, d\, f\, n12\, g^2 n5\, n4\, n10 - 2\, d\, f\, n12\, g^2 n5\, n4\, n11$$
$$- d\, f\, n13\, g^2 n5\, n4\, n10 - d\, f\, n13\, g^2 n5\, n4\, n11 - d\, f\, n14\, g^2 n5\, n4\, n10 - d\, f\, n14\, g^2 n5\, n4\, n11$$
$$+ 2\, d\, n2\, f\, n14\, g^2 n9\, n10 + 2\, d\, n2\, f\, n14\, g^2 n9\, n11 + 4\, d\, n2\, f\, n12\, g^2 n9\, n10$$
$$+ 4\, d\, n2\, f\, n12\, g^2 n9\, n11 + 2\, d\, n2\, f\, n13\, g^2 n9\, n10 + 2\, d\, n2\, f\, n13\, g^2 n9\, n11 - n11^2 g^2 - n10^2 g^2$$
$$- 6\, d\, n2\, f\, n11\, g^2 n10^2 + d\, n6\, n4\, f\, g^2 n10^2 + d\, n6\, n4\, f\, g^2 n11^2)\, th2^2 + (-3\, d^2 n2^2 f\, n11\, g^2 n10^2$$
$$- 3\, d^2 n2^2 f\, n11^2 g^2 n10 + 4\, d^2 n2\, f\, n11^2 g^2 n5^2 + 4\, d^2 n2\, f\, n10^2 g^2 n5^2 - 2\, d^2 n2^2 g^2 n10$$
$$+ 4\, d^2 n2\, g^2 n5^2 + 4\, d\, g^2 n5^2 n10 + 4\, d\, g^2 n5^2 n11 - 2\, d\, n2\, n10^2 g^2 - 2\, d\, n2\, n11^2 g^2$$
$$- 2\, d^2 n2^2 g^2 n11 + 2\, d^2 n2\, n6\, n4\, f\, g^2 n10\, n11 - 2\, d^2 n2\, n6\, n5\, f\, g^2 n9\, n10$$

$$-2\,d^2\,n2\,n6\,n5\,f\,g^2\,n9\,n11-d^2\,n2^2\,f\,n10^3\,g^2-4\,d^2\,f\,n11\,g^2\,n5^4-4\,d^2\,f\,n10\,g^2\,n5^4$$
$$-d^2\,n2^2\,f\,n11^3\,g^2-4\,d\,n2\,n11\,g^2\,n10+4\,d^2\,n6\,n5^3\,f\,g^2\,n9+4\,d^2\,n7\,n5^3\,f\,g^2\,n9$$
$$+4\,d^2\,f\,n12\,g^2\,n5^3\,n4+2\,d^2\,f\,n13\,g^2\,n5^3\,n4+2\,d^2\,f\,n14\,g^2\,n5^3\,n4-2\,d^2\,n6\,n4\,f\,g^2\,n5^2\,n10$$
$$-2\,d^2\,n6\,n4\,f\,g^2\,n5^2\,n11+2\,d^2\,n2^2\,f\,n12\,g^2\,n9\,n10+2\,d^2\,n2^2\,f\,n12\,g^2\,n9\,n11$$
$$+d^2\,n2^2\,f\,n13\,g^2\,n9\,n10+d^2\,n2^2\,f\,n13\,g^2\,n9\,n11+d^2\,n2^2\,f\,n14\,g^2\,n9\,n10$$
$$+d^2\,n2^2\,f\,n14\,g^2\,n9\,n11+8\,d^2\,n2\,f\,n11\,g^2\,n5^2\,n10+d^2\,n2\,n7\,n4\,f\,g^2\,n10^2$$
$$+d^2\,n2\,n7\,n4\,f\,g^2\,n11^2-4\,d^2\,n2\,f\,n12\,g^2\,n9\,n5^2-2\,d^2\,n2\,f\,n13\,g^2\,n9\,n5^2$$
$$-2\,d^2\,n2\,f\,n14\,g^2\,n9\,n5^2+d^2\,n2\,n6\,n4\,f\,g^2\,n11^2+d^2\,n2\,n6\,n4\,f\,g^2\,n10^2$$
$$+2\,d^2\,n2\,n7\,n4\,f\,g^2\,n10\,n11-2\,d^2\,n2\,n7\,n5\,f\,g^2\,n9\,n10-2\,d^2\,n2\,n7\,n5\,f\,g^2\,n9\,n11$$
$$-2\,d^2\,n2\,f\,n12\,g^2\,n5\,n4\,n10-2\,d^2\,n2\,f\,n12\,g^2\,n5\,n4\,n11-d^2\,n2\,f\,n13\,g^2\,n5\,n4\,n10$$
$$-d^2\,n2\,f\,n13\,g^2\,n5\,n4\,n11-d^2\,n2\,f\,n14\,g^2\,n5\,n4\,n10-d^2\,n2\,f\,n14\,g^2\,n5\,n4\,n11$$
$$-2\,d^2\,n7\,n4\,f\,g^2\,n5^2\,n11-2\,d^2\,n7\,n4\,f\,g^2\,n5^2\,n10)\,th2-d^2\,n2^2\,n10^2\,g^2-2\,d^2\,n2^2\,n11\,g^2\,n10$$
$$+4\,d^2\,n2\,g^2\,n5^2\,n10+4\,d^2\,n2\,g^2\,n5^2\,n11-4\,d^2\,n5^4\,g^2-d^2\,n2^2\,n11^2\,g^2]$$

```
> degree(Thetarels3red[2],th2); degree(Thetarels3red[3],th2);
  coeff(Thetarels3red[2],th2,4); coeff(Thetarels3red[3],th2,4);
```

$$4$$
$$4$$
$$-n5$$
$$d$$

We obtain a degree 3 polynomial by canceling the lead terms of the two degree 4 polynomials above.

```
> b3:=Reduce(collect(subs(rules1,simplify(coeff(Thetarels3red[2],th2
  ,4)*Thetarels3red[3]-coeff(Thetarels3red[3],th2,4)*Thetarels3red[2
  ])),th2),SandNbasis,tdeg(g,f,d,u,v,t));
```

$$b3:=-4\,th2^2\,d^2\,t\,g+th2\,d^3\,v\,t-th2\,d^3\,u\,t+th2^2\,d^3\,g-4\,th2^3\,t^2\,f\,g+4\,th2^2\,f^2\,v\,t$$
$$+(4\,th2-4\,th2^2)\,g\,d\,t^2+(6\,th2^2-4\,th2^3)\,g\,f\,t+(2\,th2^2+2\,th2^3+6\,th2)\,g\,d\,t$$
$$+(-th2^2+th2-th2^3)\,d^2\,v\,t+(th2^3+3\,th2-3\,th2^2)\,d^2\,u\,t+(2\,th2^2-4\,th2-2\,th2^3)\,d\,v\,t$$
$$+(-2\,th2^3+2\,th2^2)\,d\,u\,t+(-4\,th2^3+4\,th2^2)\,f\,v\,t+(th2+4\,th2^2-3\,th2^3)\,g+(th2^2-th2^3)\,v$$
$$+(th2^2-th2^3)\,u+(2\,th2^3-2\,th2^2+3\,th2)\,d\,g+(th2^2-th2+th2^3)\,f\,g$$
$$+(-2\,th2+6\,th2^2-8\,th2^3)\,t\,g+(th2-th2^2+th2^3)\,f^2\,u+(th2+3\,th2^2+th2^3)\,f^2\,v$$
$$+(-4\,th2^3-4\,th2+4\,th2^2)\,g\,t^2+(-2\,th2+2\,th2^2-2\,th2^3)\,d^2\,v+(1+th2)\,d^3\,v$$
$$+(-3\,th2^3-1+th2+3\,th2^2)\,d\,u+(3\,th2^2-3\,th2-3\,th2^3-1)\,d\,v+(8\,th2^2-th2-6\,th2^3)\,f\,v$$
$$+(4\,th2^2-2\,th2^3-th2)\,f\,u+(1-th2)\,d^3\,u+(-th2-2\,th2^2)\,g\,d^2-2\,f^2\,th2^2\,g$$

We see that b3 is not identically 0, since the constant term is not 0.

```
> factor(coeff(b3,th2,0));
```

$$d\,(d-1)\,(1+d)\,(u+v)$$

We elminiate the fusion multiplicities from the theta_i relations.

```
> ThetarelsElim:=subs(rules1,Thetarels);
```

$$ThetarelsElim := \left[-th1^2 f - 1 - \left(-\frac{5}{2} v^2 + u v + \frac{3}{2} u^2 + 2 t - \frac{1}{2} v^4 + v^3 u + t v^2 - u^3 v - 2 v u t + \frac{1}{2} u^4 + u^2 t\right) d\, th1 - (2 t - v^2 + u^2) f\, th2 - 2 \left(\frac{3}{2} v + \frac{1}{2} u - v t + u t + \frac{1}{2} v^3 - \frac{1}{2} v^2 u - \frac{1}{2} u^2 v + \frac{1}{2} u^3\right) g\, th3, -th1\, th2 - (2 t - v^2 + u^2) d\, th1 - (2 t + 1) f\, th2 - 2 (u + v) g\, th3, th1\, th3\, g - \left(\frac{3}{2} v + \frac{1}{2} u - v t + u t + \frac{1}{2} v^3 - \frac{1}{2} v^2 u - \frac{1}{2} u^2 v + \frac{1}{2} u^3\right) d\, th1 - (u + v) f\, th2 - \left(\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}\right) g\, th3 - \left(-\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}\right) g\, th3, -th3\, th2\, g - (u + v) d\, th1 - (v t + 2 v - u t) f\, th2 - t\, g\, th3 - (1 + t) g\, th3, th2^2 d - 1 - (2 t + 1) d\, th1 - (t v^2 + 2 v^2 - 2 u v - 2 v u t + 1 + 2 t + u^2 t) f\, th2 - 2 (v t + 2 v - u t) g\, th3, th3^2 f - th3^2 d - th3^2 - 1 - d\, th1 \left(\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}\right) - d\, th1 \left(-\frac{1}{2} + t + \frac{u^2}{2} - \frac{v^2}{2}\right) - f\, th2 (1 + t) - th2\, t\, f - 3 g\, th3\, v - g\, th3\, u\right]$$

```
> with(numtheory):
```

We next proceed to eliminate all but the desired possibility: t=0. For each case, we assume th2 satisfies a cyclotomic polynomial. If we trust maple to factor the polynomials irreducibly, less work is needed since only linear factors matter. We prefer to have independent confirmation that there are no integer solutions.

If either th1 or th2 is -1, get no solutions.

```
> factor(Basis(‘union‘(convert(SandNrels,set),convert(ThetarelsElim,
  set),{(th2+1)}),plex(d,f,g,th1,th2,th3,u,v,t))[1]);
```

$$(2 t + 1) (6 t + 1) (4 t^4 + 8 t^3 + 5 t^2 + t - 1)$$

If th2 is 1 then get no solutions. (check modulo 2).

```
factor(Basis(‘union‘(convert(SandNrels,set),convert(ThetarelsElim,se
t),{(th2-1)}),plex(d,th3,f,g,th2,th1,u,v,t)))[1];
```

$$(2 t + 1) (6 t + 1) (4 t^6 - 6 t^5 + 8 v^2 t^4 + 9 t^4 - 28 t^3 v^2 - 4 t^3 - 36 v^2 t^2 + 3 t^2 + 16 v^4 t^2 + 32 v^4 t + 8 t v^2 + 1 + 16 v^4 + 8 v^2) (64 v^4 + 104 v^2 + 49 - 128 t v^2 - 140 t + 32 v^2 t^2 + 156 t^2 - 80 t^3 + 16 t^4)$$

```
> factor(‘mod‘(simplify(%/((2*t+1)*(6*t+1))),4));
```

$$(t^2 - t + 1) (2 t^3 + 3 t^2 + t + 1)$$

If th2 is a 4th root of 1, no solutions.

```
factor(Basis(‘union‘(convert(SandNrels,set),convert(ThetarelsElim,se
t),{th2^2+1}),plex(d,th3,f,g,th2,th1,u,v,t))[1]);
```

$$(2 t + 1) (4 t^2 - 4 t - 1) (4 t^6 + 12 t^5 + 7 t^4 - 6 t^3 - 11 t^2 - 6 t - 1)$$

```
> factor(‘mod‘(%,2));
```

$$(t^2+t+1)(t^2-t+1)$$

If th2 is a 3rd root of unity, no solutions

```
> factor(Basis(`union`(convert(SandNrels,set),convert(ThetarelsElim,
  set),{cyclotomic(3,th2)}),plex(d,th3,f,g,th2,th1,u,v,t))[1]);
```

$$-(1+7t+16t^2+18t^3+9t^4)(4t^2+2t+1-8v^2)$$

```
> `mod`(4*t^2+2*t+1-8*v^2,4);
```

$$2t+1$$

th2^6=1, no solutions.

```
> factor(Basis(`union`(convert(SandNrels,set),convert(ThetarelsElim,
  set),{cyclotomic(6,th2)}),plex(d,th3,f,g,th2,th1,u,v,t))[1]);
```

$$(8t^2-4t-1)(128t^8+512t^7+904t^6+920t^5+573t^4+210t^3+36t^2-t-1)$$

If th2 is a 7th root of 1 then t=0, which implies u=1,v=0 as expected.

```
> factor(Basis(`union`(convert(SandNrels,set),convert(ThetarelsElim,
  set),{cyclotomic(7,th2)}),plex(d,th3,f,g,th2,th1,u,v,t)))[1];
```

$$t(1+t)(8t^3+4t^2-4t-1)(8t^3-12t^2-8t-1)(4096t^{18}+36864t^{17}+122880t^{16}+147456t^{15}$$
$$-224000t^{14}-1223936t^{13}-2439296t^{12}-2877952t^{11}-1900768t^{10}+7200t^9+1639832t^8$$
$$+2181536t^7+1753627t^6+998977t^5+417004t^4+126225t^3+26441t^2+3446t+211)$$

```
> `mod`(4096*t^18+36864*t^17+122880*t^16+147456*t^15-224000*t^14-122
  3936*t^13-2439296*t^12-2877952*t^11-1900768*t^10+7200*t^9+1639832*
  t^8+2181536*t^7+1753627*t^6+998977*t^5+417004*t^4+126225*t^3+26441
  *t^2+3446*t+211,2);
```

$$t^6+t^5+t^3+t^2+1$$

If th2 is a 9th root of 1, no integer solutions.

```
> factor(Basis(`union`(convert(SandNrels,set),convert(ThetarelsElim,
  set),{cyclotomic(9,th2)}),plex(d,th3,f,g,th2,th1,u,v,t)))[1];
```

$$(24t^3-6t-1)(8t^3-6t-1)(-9t-1+63742464t^{20}+1327104t^{23}-3422979t^{10}$$
$$+150947712t^{18}+113716224t^{19}+3881448t^8+91239696t^{16}+174t^2+13848192t^{15}$$
$$-48148515t^{12}+2085462t^7+197052t^5-63688896t^{13}+110592t^{24}+7464960t^{22}+3752982t^9$$
$$+3848t^3-22098258t^{11}+35046t^4+761903t^6-45467136t^{14}+26155008t^{21}+145846656t^{17})$$

```
> factor(`mod`(%,2));
```

$$(t^6-t^3+1)(t^6+t^4+t^3+t+1)$$

14th roots of unity slightly trickier. We use the rational-root theorem.

```
> factor(Basis(`union`(convert(SandNrels,set),convert(ThetarelsElim,
  set),{cyclotomic(14,th2)}),plex(d,th3,f,g,th2,th1,u,v,t)))[1];
```

$$(8t^3+4t^2-4t-1)(64t^6+192t^5-208t^4-64t^3+32t^2+12t+1)(232t+8-5270784t^{20}$$
$$+49152t^{23}-17853112t^{10}-61902848t^{18}-22430208t^{19}+6118813t^8-198956704t^{16}$$
$$+3114t^2-250715392t^{15}-133342745t^{12}+4196216t^7+608816t^5-207623648t^{13}+4096t^{24}$$
$$+147456t^{22}+2134759t^9+25751t^3-63466934t^{11}+146809t^4+1872713t^6-254526880t^{14}$$
$$-450560t^{21}-125888256t^{17})$$

```
> qq:=232*t+8-5270784*t^20+49152*t^23-17853112*t^10-61902848*t^18-22
  430208*t^19+6118813*t^8-198956704*t^16+3114*t^2-250715392*t^15-133
```

```
342745*t^12+4196216*t^7+608816*t^5-207623648*t^13+4096*t^24+147456
*t^22+2134759*t^9+25751*t^3-63466934*t^11+146809*t^4+1872713*t^6-2
54526880*t^14-450560*t^21-125888256*t^17;
```

$$qq := 232\,t + 8 - 5270784\,t^{20} + 49152\,t^{23} - 17853112\,t^{10} - 61902848\,t^{18} - 22430208\,t^{19} + 6118813\,t^{8} - 198956704\,t^{16} + 3114\,t^{2} - 250715392\,t^{15} - 133342745\,t^{12} + 4196216\,t^{7} + 608816\,t^{5} - 207623648\,t^{13} + 4096\,t^{24} + 147456\,t^{22} + 2134759\,t^{9} + 25751\,t^{3} - 63466934\,t^{11} + 146809\,t^{4} + 1872713\,t^{6} - 254526880\,t^{14} - 450560\,t^{21} - 125888256\,t^{17}$$

```
> ifactor(coeff(qq,t,24));
```

$$(2)^{12}$$

```
> ifactor(coeff(qq,t,0));
```

$$(2)^{3}$$

```
> seq(subs({t=2^i},qq),i=0..3);
```

-1327120136, -77676358513144, -14361851015873186552, 443076651799823026990489O4

Finally, if th2 is an 18th root of 1, no solutions.

```
> factor(Basis(‘union‘(convert(SandNrels,set),convert(ThetarelsElim,
set),{cyclotomic(18,th2)}),plex(d,th3,f,g,th2,th1,u,v,t)))[1];
```

$$(24\,t^{3} - 6\,t - 1)\,(64\,t^{6} - 192\,t^{4} - 32\,t^{3} + 36\,t^{2} + 12\,t + 1)\,(45\,t + 1 + 313344\,t^{20} + 49152\,t^{23} + 50644665\,t^{10} - 7218688\,t^{18} - 2070528\,t^{19} + 14180562\,t^{8} - 8066160\,t^{16} + 870\,t^{2} + 9663616\,t^{15} + 76832805\,t^{12} + 5467878\,t^{7} + 406122\,t^{5} + 64982928\,t^{13} + 4096\,t^{24} + 239616\,t^{22} + 29733882\,t^{9} + 9826\,t^{3} + 69839694\,t^{11} + 74292\,t^{4} + 1684269\,t^{6} + 38219568\,t^{14} + 563200\,t^{21} - 11736576\,t^{17})$$

```
> ‘mod‘(%,2);
```

$$t + 1 + t^{10} + t^{12} + t^{6}$$

```
>
```