This file contains the computations relevant to the paper "On the classification of non-self-dual modular categories" by Hong and Rowell. Thus covers the case with Galois group (01),(34), b=0, a=-g and y=f.

```
> with(LinearAlgebra): with(Groebner):
```

Using the Galois argument assuming that (01)(34), the orthogonality relations imply z=(d-1) and h1=-1/2(d+1).

```
> S1:=subs({x=1,y=f,a=-g,b=0},Matrix(5,5,[[1,d,f,g,g],[d,x,y,a,a],[f
  ,y,z,b,b],[g,a,b,h1+I*h2,h1-I*h2],[g,a,b,h1-I*h2,h1+I*h2]]));
```

$$S1 := \begin{bmatrix} 1 & d & f & g & g \\ d & 1 & f & -g & -g \\ f & f & z & 0 & 0 \\ g & -g & 0 & h1+h2\,I & h1-h2\,I \\ g & -g & 0 & h1-h2\,I & h1+h2\,I \end{bmatrix}$$

```
> C:=Matrix(5,5,[[1,0,0,0,0],[0,1,0,0,0],[0,0,1,0,0],[0,0,0,0,1],[0,
  0,0,1,0]]);
```

$$C := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

These relations describe the condition that S^2 is proportiional to the "charge conjugation matrix."

```
> Srels:=factor(convert(evalm(S1^2-K^2*C),set));
```

$$Srels := \{0, f\,(1+d+z), (-1-2\,h1+d)\,g, -(-1-2\,h1+d)\,g, 2\,d+f^2-2\,g^2, 2\,f^2+z^2-K^2,$$
$$2\,g^2+2\,h1^2-2\,h2^2, 2\,g^2+2\,h1^2+2\,h2^2-K^2, 1+d^2+f^2+2\,g^2-K^2\}$$

```
> factor(Basis(‘union‘(Srels,{1+d+z,-1-2*h1+d}),lexdeg([K,d,f,g,z],[
  h2,h1])));
```

$$[2\,h1+2+z, -1-2\,h1+d, g^2+h1^2-h2^2, 2+4\,h1-2\,h2^2+2\,h1^2+f^2, (K-2\,h2)\,(K+2\,h2)]$$

```
> s1rules:=solve({2+z+2*h1, -1-2*h1+d},{z,h1});
```

$$s1rules := \{h1 = -\frac{1}{2}+\frac{d}{2}, z = -d-1\}$$

```
> S:=subs(s1rules,S1);
```

$$S := \begin{bmatrix} 1 & d & f & g & g \\ d & 1 & f & -g & -g \\ f & f & -d-1 & 0 & 0 \\ g & -g & 0 & \frac{d}{2}-\frac{1}{2}+h2\,I & \frac{d}{2}-\frac{1}{2}-h2\,I \\ g & -g & 0 & \frac{d}{2}-\frac{1}{2}-h2\,I & \frac{d}{2}-\frac{1}{2}+h2\,I \end{bmatrix}$$

These relations come down to a single relation among the d,f and g (the other allow elimination of K, h1 and h2 and z).

```
> orthrel:={2*d+f^2-2*g^2};
```

$$orthrel := \{2\,d + f^2 - 2\,g^2\}$$

Observe that if d is an integer, then d=1 since (01) interchanges d and 1/d. this in turn implies that f is an integer dividing (d+1)=2, hence f=1 or 2, so that g^2=(2d+f^2)/2=(2+1)/2 or (2+4)/2. It must be the latter, since g^2 is an algebraic integer. So g=sqrt(3), d=1 and f=2 is the only way d is an integer. Note that since (01) interchanges g and -g/d, g cannot be an integer, and if f is an integer f=f/d so that d=1. We must consider this case eventually.

We use the various symmetries of the N_{i,j}^k to write down the fusion matrices in terms of just 14 variables. Note that M4 is just the transpose of M3.

```
> M1:=Matrix([[0,1,0,0,0],[1,n1,n2,n3,n3],[0,n2,n4,n5,n5],[0,n3,n5,n6,n7],[0,n3,n5,n7,n6]]);
```

$$M1 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & n1 & n2 & n3 & n3 \\ 0 & n2 & n4 & n5 & n5 \\ 0 & n3 & n5 & n6 & n7 \\ 0 & n3 & n5 & n7 & n6 \end{bmatrix}$$

```
> M2:=Matrix([[0,0,1,0,0],[0,n2,n4,n5,n5],[1,n4,n8,n9,n9],[0,n5,n9,n10,n11],[0,n5,n9,n11,n10]]);
```

$$M2 := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & n2 & n4 & n5 & n5 \\ 1 & n4 & n8 & n9 & n9 \\ 0 & n5 & n9 & n10 & n11 \\ 0 & n5 & n9 & n11 & n10 \end{bmatrix}$$

```
> M3:=Matrix([[0,0,0,0,1],[0,n3,n5,n7,n6],[0,n5,n9,n11,n10],[1,n6,n10,n12,n13],[0,n7,n11,n14,n12]]);
```

$$M3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & n3 & n5 & n7 & n6 \\ 0 & n5 & n9 & n11 & n10 \\ 1 & n6 & n10 & n12 & n13 \\ 0 & n7 & n11 & n14 & n12 \end{bmatrix}$$

The matrices must commute, giving a set of diophantine equations.

```
> comrels:=‘minus‘(‘union‘(convert(evalm(M1&*M2-M2&*M1),set),convert
  (evalm(M1&*M3-M3&*M1),set),convert(evalm(M2&*M3-M3&*M2),set),conve
  rt(evalm(M3&*Transpose(M3)-Transpose(M3)&*M3),set)),{0});
```

$comrels := \{n12 - n13, n13 - n12, n3\ n6 + n5\ n10 + n7\ n13 - n3\ n7 - n5\ n11 - n7\ n14,$

$n3\ n7 + n5\ n11 + n7\ n14 - n3\ n6 - n5\ n10 - n7\ n13,$

$n6\ n5 + n9\ n10 + n11\ n13 - n7\ n5 - n9\ n11 - n11\ n14,$

$n7\ n5 + n9\ n11 + n11\ n14 - n6\ n5 - n9\ n10 - n11\ n13, 1 + n6^2 + n10^2 + n13^2 - n7^2 - n11^2 - n14^2,$

$1 + n1\ n4 + n2\ n8 + 2\ n3\ n9 - n2^2 - n4^2 - 2\ n5^2, n2^2 + n4^2 + 2\ n5^2 - 1 - n1\ n4 - n2\ n8 - 2\ n3\ n9,$

$n3^2 + n5^2 + 2\ n7\ n6 - n1\ n7 - n2\ n11 - n3\ n12 - n3\ n14,$

$n5^2 + n9^2 + 2\ n11\ n10 - n4\ n7 - n8\ n11 - n9\ n12 - n9\ n14,$

$n7^2 + n11^2 + n14^2 - 1 - n6^2 - n10^2 - n13^2,$

$n1\ n7 + n2\ n11 + n3\ n12 + n3\ n14 - n3^2 - n5^2 - 2\ n7\ n6,$

$n4\ n7 + n8\ n11 + n9\ n12 + n9\ n14 - n5^2 - n9^2 - 2\ n11\ n10,$

$n1\ n5 + n2\ n9 + n3\ n10 + n3\ n11 - n3\ n2 - n5\ n4 - n6\ n5 - n7\ n5,$

$n2\ n5 + n4\ n9 + n5\ n10 + n5\ n11 - n3\ n4 - n5\ n8 - n6\ n9 - n7\ n9,$

$n2\ n6 + n4\ n10 + n5\ n13 + n5\ n12 - n3\ n5 - n5\ n9 - n6\ n10 - n7\ n11,$

$n2\ n7 + n4\ n11 + n5\ n12 + n5\ n14 - n3\ n5 - n5\ n9 - n6\ n11 - n7\ n10,$

$n3\ n2 + n5\ n4 + n6\ n5 + n7\ n5 - n1\ n5 - n2\ n9 - n3\ n10 - n3\ n11,$

$n3\ n4 + n5\ n8 + n6\ n9 + n7\ n9 - n2\ n5 - n4\ n9 - n5\ n10 - n5\ n11,$

$n3\ n5 + n5\ n9 + n6\ n10 + n7\ n11 - n2\ n6 - n4\ n10 - n5\ n13 - n5\ n12,$

$n3\ n5 + n5\ n9 + n6\ n11 + n7\ n10 - n2\ n7 - n4\ n11 - n5\ n12 - n5\ n14,$

$n3\ n6 + n5\ n10 + n6\ n13 + n7\ n12 - n3\ n7 - n5\ n11 - n6\ n12 - n7\ n14,$

$n3\ n7 + n5\ n11 + n6\ n12 + n7\ n14 - n3\ n6 - n5\ n10 - n6\ n13 - n7\ n12,$

$n6\ n5 + n9\ n10 + n10\ n13 + n11\ n12 - n7\ n5 - n9\ n11 - n10\ n12 - n11\ n14,$

$n7\ n5 + n9\ n11 + n10\ n12 + n11\ n14 - n6\ n5 - n9\ n10 - n10\ n13 - n11\ n12,$

$1 + n1\ n6 + n2\ n10 + n3\ n13 + n3\ n12 - n3^2 - n5^2 - n6^2 - n7^2,$

$1 + n4\ n6 + n8\ n10 + n9\ n13 + n9\ n12 - n5^2 - n9^2 - n10^2 - n11^2,$

$n3^2 + n5^2 + n6^2 + n7^2 - 1 - n1\ n6 - n2\ n10 - n3\ n13 - n3\ n12,$

$$n5^2+n9^2+n10^2+n11^2-1-n4\,n6-n8\,n10-n9\,n13-n9\,n12\}$$

```
> indets(comrels);
```

$$\{n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9\}$$

The commutation relations alone leave 6 degrees of freedom.

```
> HilbertDimension(comrels,tdeg(n1, n10, n11, n12, n13, n14, n2, n3,
  n4, n5, n6, n7, n8, n9));
```

$$6$$

The Characteristic polynomials of M1,M2 and M3 are computed in two ways. First from the knowledge of the eigenvalues.

```
> ch1:=collect(expand(product('(X-S[2,i]/S[1,i])',i=1..5)),X);
  ch2:=collect(expand(product('(X-S[3,i]/S[1,i])',i=1..5)),X);
  ch3:=collect(expand(product('(X-S[4,i]/S[1,i])',i=1..5)),X);
```

$$ch1 := -1+X^5+\left(-\frac{1}{d}+1-d\right)X^4+\left(-\frac{1}{d}-d\right)X^3+\left(\frac{1}{d}+d\right)X^2+\left(\frac{1}{d}-1+d\right)X$$

$$ch2 := X^5+\left(\frac{1}{f}-\frac{f}{d}+\frac{d}{f}-f\right)X^4+\left(-2-d+\frac{f^2}{d}-\frac{1}{d}\right)X^3+\left(f+\frac{f}{d}\right)X^2$$

$$ch3 := X^5+\left(-\frac{d}{g}+\frac{g}{d}+\frac{1}{g}-g\right)X^4+\left(\frac{h2^2}{g^2}-\frac{g^2}{d}-2+\frac{d^2}{4\,g^2}+\frac{1}{d}+d+\frac{1}{4\,g^2}-\frac{d}{2\,g^2}\right)X^3$$
$$+\left(\frac{3\,d}{4\,g}+g-\frac{h2^2}{g}-\frac{3}{4\,g}+\frac{1}{4\,d\,g}+\frac{h2^2}{d\,g}-\frac{d^2}{4\,g}-\frac{g}{d}\right)X^2+\left(\frac{1}{2}-\frac{1}{4\,d}-\frac{d}{4}-\frac{h2^2}{d}\right)X$$

```
>
```

set m=(d+1)/f which in an integer.

```
> simplify({subs({m=(d+1)/f},coeff(ch2,X,4)+coeff(ch2,X,2)-m),subs({
  m=(d+1)/f},subs({X=-m},ch2))});
```

$$\{0\}$$

The form of these polynomials imply further relations.

```
> simplify({coeff(ch1,X,0)+1,coeff(ch2,X,0),coeff(ch2,X,1),coeff(ch3
  ,X,0),coeff(ch1,X,4)+coeff(ch1,X,1),coeff(ch1,X,3)+coeff(ch1,X,2)}
  );
```

$$\{0\}$$

```
> factor(ch1); factor(ch2); factor(ch3);
```

$$\frac{(X-1)\,(X+1)^2\,(d\,X-1)\,(X-d)}{d}$$

$$\frac{X^2\,(X-f)\,(d+1+f\,X)\,(d\,X-f)}{d\,f}$$

$$\frac{X\,(X-g)\,(d\,X+g)\,(4\,X^2\,g^2-4\,g\,X\,d+4\,g\,X-2\,d+1+d^2+4\,h2^2)}{4\,d\,g^2}$$

```
> p1:=CharacteristicPolynomial(M1,X):
  p2:=CharacteristicPolynomial(M2,X):
  p3:=CharacteristicPolynomial(M3,X):
> factor(p1); factor(p2);
```

$(-n6+n7+X)\,(2\,n5^2-n4\,n7-n4\,n6-X^3\,n4-X^3\,n6-2\,X^2\,n5^2+X^2\,n6\,n4-X^2+X^4+X\,n4$
$+X^2\,n1\,n4+X^2\,n1\,n6+X^2\,n4\,n7+X^2\,n1\,n7+2\,X\,n1\,n5^2+2\,X\,n3^2\,n4+X\,n6\,n2^2+X\,n2^2\,n7$
$+X\,n6+X\,n7-4\,X\,n2\,n3\,n5-X\,n6\,n1\,n4-X\,n1\,n4\,n7-X^3\,n7-X^3\,n1-2\,X^2\,n3^2-X^2\,n2^2)$
$(X-n10+n11)\,(2\,n5^2-n2\,n11-n2\,n10-2\,X^2\,n5^2-X^3\,n10-2\,X^2\,n9^2-X^3\,n8+X\,n10$
$+X^2\,n10\,n8-X^2+X^4+X^2\,n8\,n11+X^2\,n2\,n10+X^2\,n2\,n8+X^2\,n2\,n11+2\,X\,n2\,n9^2+X\,n10\,n4^2$
$+X\,n4^2\,n11+X\,n11-4\,X\,n4\,n5\,n9-X\,n10\,n2\,n8-X\,n2\,n8\,n11-X^3\,n11-X^3\,n2-X^2\,n4^2$
$+X\,n2+2\,X\,n5^2\,n8)$

Observe that the only linear terms of p1 are (X-1) and (X+1), while the linear terms of p2 are X and (X+m) (provided d is not integral)
we may use this later, but for now we avoid using m.

```
> solve(X+n11-n10=(X+m)); solve(X+n11-n10=X);
```

$$\{X=X,\,m=-n10+n11,\,n10=n10,\,n11=n11\}$$

$$\{X=X,\,n10=n11,\,n11=n11\}$$

```
> mlinrels:={(n10-n11)*(n10-n11+m)};
```

$$mlinrels:=\{(n10-n11)\,(m+n10-n11)\}$$

```
> linrels:={(n6-n7)^2-1};
```

$$linrels:=\{(n6-n7)^2-1\}$$

The relations implied by the coefficients of the Characteristic polynomials.

```
> chrels:=factor({coeff(p1,X,0)+1,coeff(p2,X,0),coeff(p2,X,1),coeff(
  p3,X,0),coeff(p1,X,4)+coeff(p1,X,1),coeff(p1,X,3)+coeff(p1,X,2)});
```

$chrels:=\{(n10-n11)\,(-2\,n5^2+n2\,n11+n2\,n10),\,n4\,n6^2-n4\,n7^2-2\,n5^2\,n6+2\,n5^2\,n7+1,$
$n7^2\,n9-2\,n11\,n5\,n7+n14\,n5^2-n14\,n3\,n9+n11^2\,n3,\,-n10^2-4\,n5\,n4\,n9\,n11+4\,n10\,n4\,n5\,n9$
$+n11^2\,n4^2+n10^2\,n2\,n8+2\,n5^2-2\,n5^2\,n8\,n10-n10^2\,n4^2+2\,n11\,n5^2\,n8-2\,n2\,n10$
$-2\,n9^2\,n2\,n10-n11^2\,n2\,n8+2\,n9^2\,n11\,n2+n11^2,\,-2\,n6-n4-n1-4\,n2\,n3\,n5\,n7$
$+4\,n2\,n3\,n5\,n6-n6^2+n7^2-2\,n1\,n5^2\,n6-2\,n4\,n6+n7^2\,n2^2-n6^2\,n2^2+n1\,n4\,n6^2$
$+2\,n1\,n5^2\,n7-n1\,n4\,n7^2+2\,n5^2+2\,n3^2\,n7\,n4-2\,n3^2\,n4\,n6,\,n1\,n4-2\,n3^2+2\,n1\,n6+2\,n4\,n6$
$-n2^2-2\,n5^2-n7^2+n6^2-1+2\,n6\,n2^2+n4\,n7^2+2\,n3^2\,n6+2\,n3^2\,n4-4\,n2\,n3\,n5-2\,n6\,n1\,n4$
$-n4\,n6^2+n4+2\,n6-n1\,n6^2-2\,n3^2\,n7+n1\,n7^2+2\,n1\,n5^2-2\,n5^2\,n7+2\,n5^2\,n6\}$

```
> nops(chrels);
```

$$6$$

these involve the integer m=z/f=(d+1)/f.

```
> intchar:=Vector([1,1,-m,0,0]):
> intcharrels:=‘union‘(convert(evalm(M1&*intchar-intchar),set),conve
  rt(evalm(M2&*intchar+m*intchar),set),convert(evalm(M3&*intchar),se
  t),convert(evalm(Transpose(M3)&*intchar),set));
```

$intcharrels:=\{$
$0,\,n1-n2\,m,\,n3-n5\,m,\,n5-n9\,m,\,n7-n11\,m,\,1+n6-n10\,m,\,n2-n4\,m+m,\,1+n4-n8\,m-m^2$
$\}$

```
> mrels:=‘union‘({coeff(p2,X,4)+coeff(p2,X,2)-m,subs({X=-m},p2)},mli
```

```
nrels,intcharrels);
```

$$
\begin{aligned}
&\mathit{mrels} := \{0, (n10 - n11)\,(m + n10 - n11), n1 - n2\,m, n3 - n5\,m, n5 - n9\,m, n7 - n11\,m,\\
&\quad 1 + n6 - n10\,m, n2 - n4\,m + m, 1 + n4 - n8\,m - m^2, -m^5 - (2\,n10 + n8 + n2)\,m^4\\
&\quad + (2\,n5^2 + 2\,n9^2 + n11^2 - n10^2 - 2\,n8\,n10 - 2\,n2\,n10 + 1 + n4^2 - n2\,n8)\,m^3 - (-n2 - 2\,n10\\
&\quad + 2\,n11\,n5^2 - 2\,n5^2\,n8 + 4\,n4\,n5\,n9 - 2\,n2\,n9^2 + 2\,n10\,n2\,n8 - 2\,n10\,n4^2 + 2\,n9^2\,n11\\
&\quad - 2\,n10\,n5^2 - 2\,n10\,n9^2 - n11^2\,n8 - n11^2\,n2 + n8\,n10^2 + n2\,n10^2)\,m^2 + (-2\,n5^2 + 2\,n2\,n10\\
&\quad + n10^2 - n11^2 - 2\,n11\,n5^2\,n8 + 4\,n5\,n4\,n9\,n11 - 4\,n10\,n4\,n5\,n9 - 2\,n9^2\,n11\,n2 + 2\,n5^2\,n8\,n10\\
&\quad + 2\,n9^2\,n2\,n10 + n11^2\,n2\,n8 - n10^2\,n2\,n8 - n11^2\,n4^2 + n10^2\,n4^2)\,m + 2\,n11\,n5^2 - 2\,n10\,n5^2\\
&\quad - n11^2\,n2 + n2\,n10^2, -n8 + 2\,n5^2\,n8 - 4\,n4\,n5\,n9 + n11^2\,n8 - 2\,n9^2\,n11 + 2\,n10\,n5^2 - n2\,n10^2\\
&\quad + 2\,n10\,n4^2 + n11^2\,n2 + 2\,n2\,n9^2 - 2\,n10\,n2\,n8 + 2\,n10\,n9^2 - n8\,n10^2 - 2\,n11\,n5^2 - m\}
\end{aligned}
$$

We take all of the relations together and process them. .

```
> allNrels:=subs({n8=u,n11=v,n13=t,n14=w},‘union‘(linrels,comrels,ch
  rels,mrels)):
> factor(Basis(allNrels,lexdeg([w, v, t, u,n1,n6, n9,n10,n12, n2,
  n3, n4, n5,n7],[m])));
```

$$
\begin{aligned}
&[n3 - n5\,m, n2 - n4\,m + m, -1 - n4 + 2\,n10\,m, -n5 + n9\,m, -n4 + 1 + 2\,n6, n1 - n4\,m^2 + m^2,\\
&\quad -1 - n4 + u\,m + m^2, t - n12, -n7 + v\,m, (2 + m^2)\,(n7 + 1)\,(1 - 2\,n7 + n4), n7\,(-2\,n10 + m + u),\\
&\quad -1 - n4 + 2\,n5^2 + 2\,n7^2 + 3\,n7 - 3\,n4\,n7,\\
&\quad n5 + n5\,n4 - 2\,n7\,n5 - 2\,n7\,w\,m + n4\,m^2\,n5 - m^2\,n5 + 2\,n7\,m\,n12 - 2\,n7\,m^2\,n5,\\
&\quad -1 - n4 + 3\,n7 + 2\,n7^2 - 3\,n4\,n7 + 2\,m^2 - 2\,n4\,m^2 + 4\,n5\,n12\,m,\\
&\quad 2\,n5\,n10 - 2\,n7\,n9 - n5\,m + 2\,n7\,n12 - 2\,n7\,w - 2\,n7\,n5\,m + m\,n5\,n4,\\
&\quad -m + n5\,n9 - 2\,n5\,n12 + n4\,m, -2\,n7\,n9 + 2\,n7\,n12 - 2\,n7\,w + n5\,u - 2\,n7\,n5\,m + m\,n5\,n4,\\
&\quad -n7\,n9 + n5\,v, -3\,m - n4\,n7\,m - 2\,n5\,n12 + n4\,m + 2\,n5\,w + n7\,m + 2\,n7^2\,m,\\
&\quad (1 - 2\,n7 + n4)\,(n4 - 3 - 2\,n7), n5 + 2\,m\,n12\,n4 - 2\,w\,m - 2\,n7\,w\,m - n5\,n4 - 3\,m^2\,n5\\
&\quad + 2\,n7\,m\,n12 - n7\,m^2\,n5 - w\,m^3 - m^4\,n5 + n12\,n4\,m^3,\\
&\quad n4\,m + m + n10 + n4\,n7\,m - n7\,m + n4\,n10 - 2\,n7\,v - 2\,n7^2\,m,\\
&\quad n9 + n4\,n9 - 2\,n7\,n9 - n5\,m + 2\,n7\,n12 - 2\,n7\,w - 2\,n7\,n5\,m + m\,n5\,n4,\\
&\quad 3\,n4\,m + 3\,m + u + 2\,n4\,n7\,m - 2\,n7\,m - 4\,n7\,v - 4\,n7^2\,m + n4\,u, v - 2\,n7\,n10 + n4\,v,\\
&\quad -n12 + w + 3\,n5\,m - 2\,n7\,n12 + 2\,n7\,w + 2\,n7\,n5\,m - m\,n5\,n4 + w\,n4 - n12\,n4,\\
&\quad -3 + n4 - 4\,n9\,n12 + 2\,n9^2 + 4\,n10^2 - 2\,u\,n10,\\
&\quad -5 - n4 + n7 + 2\,n7^2 - n4\,n7 + 8\,n10^2 + 2\,n9\,w - 4\,u\,n10 - 2\,n9\,n12,\\
&\quad 1 + n4 + 2\,n7 - 4\,v\,n10 + 2\,u\,v - 4\,n10^2 + 2\,u\,n10, -1 - n4 + 2\,n7^2 - 2\,n10^2 + n7 - n4\,n7 + 2\,v^2,\\
&\quad n5 + 2\,v\,w - 2\,v\,n12 - n5\,n4 + 2\,n7\,n5 - 2\,n9\,n10 + 2\,n9\,v,\\
&\quad n4 + n7 + 2\,w^2 + 2\,n7^2 - n4\,n7 - 2\,n12^2 - 3, n5 + 2\,n7\,n5 - n5\,n7^2 + 2\,m\,n12\,n4 - 3\,w\,m - n7\,w\,m\\
&\quad - 2\,n5\,n4 - 3\,m^2\,n5 + n7\,m\,n12 - n7\,m^2\,n5 - m\,n12 + 2\,n12\,n7^2\,m, 2\,n12 - 2\,n7\,n9 + 2\,n7\,n12
\end{aligned}
$$

$+ 2\, n7^2\, n9 - 2\, n5\, m - 2\, n7\, w - 4\, n12\, n7^2 - 3\, w\, m^2 + n7\, n5\, m - 3\, m^3\, n5 - m\, n5\, n4 + 3\, n12\, n4\, m^2$
$+ 2\, n12\, n4, -n7\, (n7 + 1)\, (-2\, n7\, m + 4\, n10 + n4\, m + m - 4\, v), -4\, n7\, n12 + n7\, m^3\, n5 + n7\, m^2\, w$
$+ 5\, n5\, m + 4\, n7\, w - 4\, n12\, n7^2 - 3\, w\, m^2 + 6\, n7\, n5\, m - m^3\, n5 + 2\, n12\, n4\, m^2 - n7^2\, n5\, m + 4\, w\, n7^2$
$- n12\, n7\, m^2 + 2\, n7^2\, m^2\, w - 2\, m\, n5\, n4 - m^2\, n12, 2\, n5 - n5\, n4 + n7\, n5 - 3\, n5\, n7^2 + n4\, n5\, n7$
$+ 2\, m\, n12\, n4 - 3\, w\, m + n7\, w\, m - m^2\, n5 - n7\, m\, n12 + n7\, m^2\, n5 - m\, n12 + 2\, n7^2\, m\, w,$
$n7\, (-3\, n4\, m - 8\, n7\, n10 + 8\, v + 5\, m + n4\, n7\, m - n7\, m - 2\, n7^2\, m + 8\, n5\, n12), n7\, n12\, n4 + n12$
$- 2\, n7\, n12 + n5\, m + n7\, w - 3\, n12\, n7^2 - w\, m^2 - m^3\, n5 + n12\, n4\, m^2 + w\, n7^2 + n12\, n4 - m\, n5\, n4$
$+ 2\, n7\, n5\, m,$
$4\, n12\, n7\, w + 2\, n4 - 5\, n7 + 2 - 3\, n7^2 + 3\, n4\, n7 + 2\, n7^3 - n4\, n7^2 - 2\, m^2 - 4\, n12^2\, n7 + 2\, n4\, m^2,$
$-2\, n7^2\, m\, w + n7\, m\, n12 - 3\, n5 - 2\, n5\, n4 + 6\, n7\, n5 + 8\, n9\, n10 - 2\, n10\, n12 + n5\, n7^2 + 8\, n10\, n7\, n9$
$+ 6\, m\, n12\, n4 - 8\, v\, n12\, n7 - 12\, n9\, v + 8\, v\, n12 + 3\, n12\, u - 4\, w\, m - n7\, w\, m + 2\, n10\, w - 2\, u\, n9$
$+ u\, w - 7\, m^2\, n5 + 4\, m\, n12 - 8\, n10\, n12\, n7 - n7\, m^2\, n5,$
$-1 - n4 - n7 + 2\, n7\, n10\, v + 2\, n7^2 + n4\, n7 - 4\, n9\, n12\, n7 - 2\, n10^2 - 2\, v\, n10 + 2\, n7\, n10^2, 2\, n7^2\, m\, w$
$- n7\, m\, n12 + 3\, n5 - 2\, n5\, n4 + 6\, n7\, n5 - 8\, n9\, n10 + 6\, n10\, n12 - n5\, n7^2 + 2\, m\, n12\, n4 + 4\, n9\, v$
$- 3\, n12\, u - 4\, w\, m + n7\, w\, m + 2\, n10\, w + 2\, u\, n9 - u\, w - m^2\, n5 - 4\, m\, n12 - 4\, n10\, n12\, n7$
$+ 4\, w\, n7\, n10 + n7\, m^2\, n5, 2\, n7^2\, m\, w - n7\, m\, n12 + 3\, n5 - 6\, n5\, n4 + 10\, n7\, n5 - 8\, n9\, n10$
$+ 10\, n10\, n12 - n5\, n7^2 + 10\, m\, n12\, n4 - 8\, v\, n12\, n7 - 4\, n9\, v + 8\, v\, n12 - 3\, n12\, u - 12\, w\, m + n7\, w\, m$
$+ 6\, n10\, w + 2\, u\, n9 - u\, w - 9\, m^2\, n5 - 4\, m\, n12 - 8\, n10\, n12\, n7 + 8\, v\, n7\, n9 + n7\, m^2\, n5, -2\, m$
$- 3\, n4\, n7\, m - 8\, n10\, n7^2 - 2\, n4\, m + 8\, n7\, v + 4\, n5\, n12 + 9\, n7\, m + n7^2\, m + 3\, n4\, n7^2\, m$
$+ 4\, n12\, n4\, n5 - 6\, n7^3\, m, -4\, n7^2\, m\, w + 2\, n7\, m\, n12 + 16\, n12^2\, n5 - 4\, n5\, n4 + 4\, n7\, n5 + 8\, n9\, n10$
$+ 2\, n10\, n12 + 2\, n5\, n7^2 - 4\, m\, n12\, n4 - 8\, v\, n12\, n7 - 4\, n9\, v + 8\, v\, n12 + n12\, u - 3\, w\, m - 2\, n7\, w\, m$
$+ 6\, n10\, w - 2\, u\, n9 - u\, w - 6\, m^2\, n5 + 11\, m\, n12 - 8\, n10\, n12\, n7 - 2\, n7\, m^2\, n5, -8\, n12\, w + 13$
$+ 13\, n4 - 2\, n7 - 13\, n7^2 - 8\, n7^2\, m^2 - 4\, n4\, n7 + 8\, n9\, n12 - 16\, n10^2 + 6\, n7^3 - 5\, n4\, n7^2 - 2\, m^4\, n4$
$+ 6\, m^2 + 16\, v\, n10 + 8\, u\, n10 - 16\, n7\, n10^2 + 2\, m^4 + 8\, n12^2\, n4 + 2\, n4\, m^2 + 4\, n12^2\, n4\, m^2$
$- 4\, n12\, w\, m^2 + 8\, n9\, n12\, n7, -16\, n10 + 4\, u - 4\, m - 3\, n4\, n7\, m - 8\, n5\, n12 + 8\, n9\, v\, n12 + 8\, n7\, v$
$- 8\, n12\, n9\, n10 + 5\, n7\, m + n4\, n7^2\, m + 7\, n7^2\, m - 2\, n7^3\, m, -4\, w\, n10\, n12 - 16\, n10 + 4\, u - 3\, m$
$+ 2\, w\, m\, n12 + 2\, u\, w\, n12 - 3\, n4\, n7\, m - 12\, n5\, n12 + 6\, m\, n12^2 - 12\, n10\, n12^2 + n4\, m + 8\, n7\, v$
$+ 8\, n12\, n9\, n10 - 4\, n12\, u\, n9 + 3\, n7\, m + 6\, n7^2\, m + 6\, n12^2\, u, -4\, n10 - 8\, v + 2\, u - 7\, m - 5\, n4\, n7\, m$
$+ 8\, n7\, n10 - 8\, n5\, n12 - n4\, m + 8\, n7\, v + 8\, u\, n10^2 + 5\, n7\, m + 10\, n7^2\, m - 16\, n10^3, -4\, n10\, n12\, u$
$+ 2\, n9 - 2\, n12 - 2\, w + 2\, n7\, n9 - 2\, n5\, m - w\, m^2 - 8\, n10^2\, n9 + 4\, u\, n10\, n9 + n7\, n5\, m - m^3\, n5$
$- m\, n5\, n4 + n12\, n4\, m^2 + 8\, n10^2\, n12, -2\, n7\, n9 + 2\, n7\, n12 + 2\, n9\, v\, n10 - n5\, m - 2\, n7\, w$
$- 2\, n10^2\, w - 2\, n10^2\, n9 - 2\, n7\, n5\, m + m\, n5\, n4 + 2\, n10^2\, n12, -44\, n10 - 24\, v + 14\, u + 4\, n10\, u^2$

$-25\,m-17\,n4\,n7\,m+24\,n7\,n10-44\,n5\,n12+8\,m\,n12^2-16\,n10\,n12^2+n4\,m+32\,n7\,v$
$+8\,n12\,n9\,n10-4\,n12\,u\,n9+17\,n7\,m+34\,n7^2\,m+8\,n12^2\,u-16\,n10^3,\ 2\,n10\,n12\,u-2\,n9$
$+2\,n12+2\,w+2\,n7\,n9-2\,n7\,n12+n5\,m+2\,n7\,w+w\,m^2-4\,n10^2\,w+n7\,n5\,m+m^3\,n5$
$-n12\,n4\,m^2-4\,n10^2\,n12+2\,u\,w\,n10,\ -4\,n10\,n12\,u+2\,m^2\,n12+6\,n9-6\,n12-6\,w+2\,n7\,n9$
$+2\,u^2\,n9+4\,n7\,n12-10\,n5\,m-4\,n7\,w-3\,w\,m^2-8\,n10^2\,n9-n7\,n5\,m-3\,m^3\,n5-2\,n12\,u^2$
$-m\,n5\,n4+3\,n12\,n4\,m^2+16\,n10^2\,n12,\ 4\,n10\,n12\,u-m^2\,n12-6\,n9+6\,n12+6\,w+6\,n7\,n9$
$-8\,n7\,n12+6\,n5\,m+8\,n7\,w+2\,w\,m^2-4\,n10^2\,w+5\,n7\,n5\,m+3\,m^3\,n5+n12\,u^2-m\,n5\,n4$
$-3\,n12\,n4\,m^2-12\,n10^2\,n12+u^2\,w,\ 24\,n12\,w-14-14\,n4-36\,n7-5\,n7^2+8\,n7^2\,m^2+40\,n4\,n7$
$-24\,n9\,n12+40\,n10^2+8\,n12^2-5\,n7^3-2\,n7^4+n4\,n7^3+11\,n4\,n7^2+8\,n7^2\,n10^2-12\,m^2$
$-56\,v\,n10-24\,u\,n10+56\,n7\,n10^2-16\,n12^2\,n4+4\,n4\,m^2-16\,n12^2\,n7^2-48\,n9\,n12\,n7,$
$-5\,n7^3\,m\,n4+24\,w\,n10\,n12+20\,m^3-40\,n10-16\,v+12\,u-38\,m-40\,w\,m\,n12-50\,n4\,n7\,m$
$+16\,n7\,n10+24\,n10\,n7^2+32\,n7\,n10^3-48\,v\,n12^2\,n7+24\,n10\,n12^2-18\,n4\,m-8\,n7\,v$
$-32\,n12\,n9\,n10+66\,n7\,m+40\,m\,n12^2\,n4+3\,n4\,n7^2\,m+55\,n7^2\,m-20\,n4\,m^3-16\,n10\,n12^2\,n7$
$+10\,n7^4\,m-11\,n7^3\,m-32\,v\,n10^2+16\,n12^2\,v,\ 4+4\,n4+3\,n7+n7^2-5\,n4\,n7-24\,n10^2$
$+4\,n12^2+2\,n7^3-16\,n12^2\,n10^2-n4\,n7^2+2\,m^2+8\,v\,n10+16\,u\,n10-8\,n7\,n10^2-2\,u^2$
$+4\,n12^2\,n4+8\,n12^2\,u\,n10+8\,n9\,n12\,n7,\ 2+2\,n4+2\,n7+3\,n7^2-6\,n4\,n7+3\,n7^3-2\,n7^4$
$-8\,n12^2\,n10^2+n4\,n7^3-n4\,n7^2+8\,v\,n10-8\,n10^2+4\,u\,n10+8\,n12\,n10^2\,w-8\,n7\,n10^2$
$+16\,n9\,n12\,n7,\ -56\,w\,n10\,n12-8\,m^3+976\,n10+224\,v-272\,u+325\,m-8\,u^3+238\,n4\,n7\,m$
$-224\,n7\,n10+736\,n5\,n12+128\,n10\,n12^4-16\,v\,n12^2\,n7-312\,m\,n12^2+664\,n10\,n12^2$
$-59\,n4\,m-528\,n7\,v-272\,n12\,n9\,n10+136\,n12\,u\,n9-64\,m\,n12^4-252\,n7\,m+16\,m\,n12^2\,n4$
$-7\,n4\,n7^2\,m-483\,n7^2\,m-296\,n12^2\,u+32\,n12^3\,u\,n9+48\,n10\,n12^2\,n7+14\,n7^3\,m$
$-64\,n12^3\,n9\,n10+64\,n10^3-48\,n12^2\,v-64\,n12^4\,u,\ 24\,n12\,w-8-8\,n4+68\,n7+126\,n7^2$
$-156\,n4\,n7-24\,n9\,n12-32\,n10^2+80\,n12^2-32\,n10^4+2\,u^4+87\,n7^3-5\,n7^4-128\,n12^2\,n10^2$
$+3\,n4\,n7^3-42\,n4\,n7^2-8\,n12^4\,m^2-36\,m^2+56\,v\,n10-16\,u\,n10+32\,n9\,n12^3\,n7$
$-32\,n10^2\,n12^2\,n7-56\,n7\,n10^2-2\,m^4+32\,u^2+56\,n12^2\,n4+8\,n12^4\,u^2-2\,n7^5+16\,n12^4\,n4$
$+20\,n4\,m^2-32\,m^2\,n12^2+32\,n12^2\,u^2-32\,n12^2\,n7^2+16\,n12^4-32\,n12^4\,n10^2+n7^4\,n4$
$+144\,n9\,n12\,n7+32\,n12^2\,v\,n10]$

m>0 an integer implies:.

```
> factor(Basis(`union`(allNrels,{1-2*n7+n4,m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5,n5*m-n7*n12+n7*w}),plex(n1,n6, n9,n12, n2, n3, n4,w, v,n10, t, u, n5,n7,m)));
```

$[n5^2+2\,n7-2\,n7^2,\ -2\,n7+m^2+u\,m,\ -(n7-1)\,(-4\,n7\,t+3\,n5\,m+n5\,u),$
$n7+n5\,m\,t-n7^2+m^2-n7\,m^2,\ -n7\,(-3\,m-u+u\,n7-2\,n5\,t+3\,n7\,m),$

$6\,n7+2\,u\,n5\,t-u^2\,n7-3\,m^2+u^2-6\,n7^2+3\,n7\,m^2,\ -n7+n10\,m,\ -n7\,(-2\,n10+m+u),$

$-n5\,(-2\,n10+m+u),\ -2\,n7+m^2+2\,u\,n10-u^2,\ -4\,n7+4\,n10^2+m^2-u^2,$

$-6\,n7\,m+3\,m+2\,n10+4\,v-u+4\,n5\,t-2\,u\,n7,\ -n5+t\,m+w\,m+n7\,n5+m^2\,n5-2\,n7\,m\,t,$

$n5\,m-n7\,t+n7\,w,\ 2\,n7\,m-2\,m-n5\,t+n5\,w,$

$-3\,n5\,m+2\,n5\,u+2\,t\,m^2+n7\,n5\,m-u^2\,t+m^3\,n5-2\,n7\,m^2\,t+u^2\,w,\ 8\,m-8\,u+32\,n10-19\,n7\,m$

$-7\,u\,n7+22\,n5\,t+2\,m^3-5\,n7^2\,m+2\,t^2\,m-n7^2\,u-4\,u\,t^2+12\,n10\,t^2-2\,n7\,m^3-8\,n7\,m\,t^2$

$+8\,t^3\,n5-4\,t^2\,u\,n7+2\,u\,w\,t,$

$n5+5\,t\,m+n7\,n5+2\,n10\,t+m^2\,n5-8\,n7\,m\,t+4\,n5\,t^2-2\,u\,n7\,t-u\,w+4\,n10\,w,$

$2\,n7-2+w^2-t^2,\ 1-2\,n7+n4,\ n3-n5\,m,\ n2-2\,n7\,m+2\,m,\ n12-t,$

$-2\,t-2\,w-3\,n5\,m+4\,n7\,t-n5\,u+2\,n9,\ -n7+1+n6,\ 2\,m^2-2\,n7\,m^2+n1]$

assuming n7=0 gives m=0.

```
> factor(Basis(‘union‘(allNrels,{1-2*n7+n4,m^2*n5+m*n12+w*m-2*n7*m*n
  12-n5+n7*n5,n5*m-n7*n12+n7*w,n7}),plex(n1,n6, n9,n12, n2, n3,
  n4,w, v,n10, t, u, n5,n7,m)));
```

$[m^2,\ n7,\ n5\,m,\ n5^2,\ u\,m,\ n5\,u,\ u^2,\ n10\,m,\ n5\,n10,\ u\,n10,\ n10^2,\ 4\,n5\,t-u+4\,v+2\,n10+3\,m,$

$w\,m-n5+t\,m,\ -2\,m-n5\,t+n5\,w,$

$4\,m-4\,u+16\,n10+11\,n5\,t+t^2\,m-2\,u\,t^2+6\,n10\,t^2+4\,t^3\,n5+u\,w\,t,$

$n5+5\,t\,m+4\,n5\,t^2+2\,n10\,t-u\,w+4\,n10\,w,\ w^2-2-t^2,\ n4+1,\ n3,\ n2+2\,m,\ n12-t,\ n9-w-t,$

$n6+1,\ n1]$

Thus we may assume n7 not zero since then we have m=0 .

```
> factor(Basis(‘union‘(allNrels,{1-2*n7+n4,m^2*n5+m*n12+w*m-2*n7*m*n
  12-n5+n7*n5,n5*m-n7*n12+n7*w,-3*m-u-2*n5*t+u*n7+3*n7*m,-2*n10+u+m}
  ),plex(n1,n6, n9,n12, n2, n3, n4,w, v,n10, t, u, n5,n7,m)));
```

$[n5^2+2\,n7-2\,n7^2,\ -2\,n7+m^2+u\,m,\ -(n7-1)\,(-4\,n7\,t+3\,n5\,m+n5\,u),$

$3\,m+u-u\,n7+2\,n5\,t-3\,n7\,m,\ 2\,n10-m-u,\ -m+2\,v-u,$

$-n5+t\,m+w\,m+n7\,n5+m^2\,n5-2\,n7\,m\,t,\ n5\,m-n7\,t+n7\,w,\ -m+u+n7\,m-u\,n7+2\,n5\,w,$

$3\,n5-2\,t\,m-n7\,n5-m^2\,n5+2\,n7\,m\,t+u\,w-u\,t,\ 2\,n7-2+w^2-t^2,\ 1-2\,n7+n4,\ n3-n5\,m,$

$n2-2\,n7\,m+2\,m,\ n12-t,\ -2\,t-2\,w-3\,n5\,m+4\,n7\,t-n5\,u+2\,n9,\ -n7+1+n6,$

$2\,m^2-2\,n7\,m^2+n1]$

finally, we solve for all but w,v,t,u.

```
> intrels:=factor(Basis(‘union‘(allNrels,{1-2*n7+n4,m^2*n5+m*n12+w*m
  -2*n7*m*n12-n5+n7*n5,n5*m-n7*n12+n7*w,-3*m-u-2*n5*t+u*n7+3*n7*m,-2
  *n10+u+m}),lexdeg([m,n1,n6, n9,n10,n12, n2, n3, n4,n5,n7],[u, v,
  t, w])));
```

$intrels := [2+t^2-w^2+2\,u\,v-4\,v^2,\ (-w+t)\,(-3\,v\,w-v\,t+u\,t+u\,w),$

$-(-w+t)\,(t^3+t^2\,w+2\,t-2\,t\,v^2-t\,w^2+2\,w\,v^2+2\,w-w^3),\ 2\,n7-2+w^2-t^2,\ v\,w-v\,t+n5,$

$-1+n4+w^2-t^2,\ 2\,n3+2\,w+t^2\,w-w^3-2\,t-t^3+t\,w^2,\ 2\,w\,t\,v+n2-t^2\,v-v\,w^2,\ n12-t,\ n10-v,$

$$t^3+t^2 w-t w^2-w^3+2\,n9+2\,w v^2-2\,t v^2,\ w^2-t^2+2\,n6,$$

$$2\,n1+4\,t w+2\,w t^3-2\,t w^3-2\,w^2+w^4-2\,t^2-t^4,\ m-2\,v+u]$$

```
> rules1:=`union`({n8=u,n11=v,n13=t,n14=w},solve({seq(intrels[i],i=4..14)}));
```

$$rules1 := \{m=-u+2\,v,\ n1=t^2-2\,t w+w^2+\frac{1}{2}t^4-w t^3+t w^3-\frac{1}{2}w^4,\ n10=v,\ n11=v,\ n12=t,$$

$$n13=t,\ n14=w,\ n2=v w^2-2\,w t v+t^2 v,\ n3=t-w+\frac{1}{2}t^3-\frac{1}{2}t^2 w-\frac{1}{2}t w^2+\frac{1}{2}w^3,$$

$$n4=1+t^2-w^2,\ n5=-v w+v t,\ n6=\frac{t^2}{2}-\frac{w^2}{2},\ n7=1+\frac{t^2}{2}-\frac{w^2}{2},\ n8=u,$$

$$n9=\frac{1}{2}w^3+\frac{1}{2}t w^2-\frac{1}{2}t^2 w-w v^2-\frac{1}{2}t^3+t v^2,\ t=t,\ u=u,\ v=v,\ w=w\}$$

```
> indets(subs(rules1,`union`(allNrels,{1-2*n7+n4,m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5,n5*m-n7*n12+n7*w,-3*m-u-2*n5*t+u*n7+3*n7*m,-2*n10+u+m})));
```

$$\{t, u, v, w\}$$

```
> agb:=factor(Basis(subs(rules1,`union`(allNrels,{1-2*n7+n4,m^2*n5+m*n12+w*m-2*n7*m*n12-n5+n7*n5,n5*m-n7*n12+n7*w,-3*m-u-2*n5*t+u*n7+3*n7*m,-2*n10+u+m})),plex( u, v, t,w )));
```

$$agb := [-(-w+t)(t^3+t^2 w+2\,t-2\,t v^2-t w^2+2\,w v^2+2\,w-w^3),$$

$$(-w+t)(-3\,v w-v t+u t+u w),\ 2+t^2-w^2+2\,u v-4\,v^2]$$

```
> M1s:=subs(rules1,M1); M2s:=subs(rules1,M2); M3s:=subs(rules1,M3);
```

$M1s :=$

$$[0,1,0,0,0]$$

$$\left[1,\ t^2-2\,t w+w^2+\frac{1}{2}t^4-w t^3+t w^3-\frac{1}{2}w^4,\ v w^2-2\,w t v+t^2 v,\right.$$

$$\left.t-w+\frac{1}{2}t^3-\frac{1}{2}t^2 w-\frac{1}{2}t w^2+\frac{1}{2}w^3,\ t-w+\frac{1}{2}t^3-\frac{1}{2}t^2 w-\frac{1}{2}t w^2+\frac{1}{2}w^3\right]$$

$$[0,\ v w^2-2\,w t v+t^2 v,\ 1+t^2-w^2,\ -v w+v t,\ -v w+v t]$$

$$\left[0,\ t-w+\frac{1}{2}t^3-\frac{1}{2}t^2 w-\frac{1}{2}t w^2+\frac{1}{2}w^3,\ -v w+v t,\ \frac{t^2}{2}-\frac{w^2}{2},\ 1+\frac{t^2}{2}-\frac{w^2}{2}\right]$$

$$\left[0,\ t-w+\frac{1}{2}t^3-\frac{1}{2}t^2 w-\frac{1}{2}t w^2+\frac{1}{2}w^3,\ -v w+v t,\ 1+\frac{t^2}{2}-\frac{w^2}{2},\ \frac{t^2}{2}-\frac{w^2}{2}\right]$$

$M2s :=$

$$[0,0,1,0,0]$$

$$[0,\ v w^2-2\,w t v+t^2 v,\ 1+t^2-w^2,\ -v w+v t,\ -v w+v t]$$

$$\left[1, 1+t^2-w^2, u, \frac{1}{2}w^3+\frac{1}{2}t w^2-\frac{1}{2}t^2 w-w v^2-\frac{1}{2}t^3+t v^2, \frac{1}{2}w^3+\frac{1}{2}t w^2-\frac{1}{2}t^2 w-w v^2-\frac{1}{2}t^3+t v^2\right]$$

$$\left[0, -v w+v t, \frac{1}{2}w^3+\frac{1}{2}t w^2-\frac{1}{2}t^2 w-w v^2-\frac{1}{2}t^3+t v^2, v, v\right]$$

$$\left[0, -v w+v t, \frac{1}{2}w^3+\frac{1}{2}t w^2-\frac{1}{2}t^2 w-w v^2-\frac{1}{2}t^3+t v^2, v, v\right]$$

$$M3s := \begin{bmatrix} 0, & 0, & 0, & 0, & 1 \\ 0, & t-w+\frac{1}{2}t^3-\frac{1}{2}t^2 w-\frac{1}{2}t w^2+\frac{1}{2}w^3, & -v w+v t, & 1+\frac{t^2}{2}-\frac{w^2}{2}, & \frac{t^2}{2}-\frac{w^2}{2} \\ 0, & -v w+v t, & \frac{1}{2}w^3+\frac{1}{2}t w^2-\frac{1}{2}t^2 w-w v^2-\frac{1}{2}t^3+t v^2, & v, & v \\ 1, & \frac{t^2}{2}-\frac{w^2}{2}, & v, & t, & t \\ 0, & 1+\frac{t^2}{2}-\frac{w^2}{2}, & v, & w, & t \end{bmatrix}$$

```
>
```

The unique positive character is psi0.

```
> psi0:=Vector([1,d,f,g,g]): psi2:=subs({m=(d+1)/f},intchar):
> psirels:=numer(‘minus‘(‘union‘(convert(evalm(M1s&*psi0-psi0[2]*psi
  0),set),convert(evalm(M2s&*psi0-psi0[3]*psi0),set),convert(evalm(M
  3s&*psi0-psi0[4]*psi0),set),convert(evalm(M1s&*psi2-psi2[2]*psi2),
  set),convert(evalm(M2s&*psi2-psi2[3]*psi2),set),convert(evalm(M3s&
  *psi2-psi2[4]*psi2),set)),{0}));
```

$$psirels := \{2 f+t^2 f-w^2 f-2 v d-2 v, 2+d t^2-d w^2+2 v f+4 t g-2 g^2,$$
$$2 d+d t^2-d w^2+2 v f+2 w g+2 t g-2 g^2, f v w^2-2 f w t v+f t^2 v-d t^2-t^2+d w^2+w^2,$$
$$2 f^2+t^2 f^2-w^2 f^2-u f d-u f-d^2-2 d-1,$$
$$d v w^2-2 d w t v+d t^2 v+f+t^2 f-w^2 f-2 g v w+2 g v t-d f,$$
$$2 t f-2 w f+f t^3-f t^2 w-f t w^2+f w^3+2 d v w+2 v w-2 d v t-2 v t,$$
$$-2 d v w+2 d v t+f w^3+f t w^2-f t^2 w-2 f w v^2-f t^3+2 f t v^2+4 v g-2 f g,$$
$$1+d+d t^2-d w^2+u f+g w^3+g t w^2-g t^2 w-2 g w v^2-g t^3+2 g t v^2-f^2,$$
$$2 d t-2 d w+t^3 d-t^2 w d-t w^2 d+w^3 d-2 v w f+2 v t f+2 g t^2-2 g w^2+2 g-2 d g, 2 t^2 f$$
$$-4 t w f+2 w^2 f+t^4 f-2 w t^3 f+2 t w^3 f-w^4 f-2 d t^2 v-2 t^2 v+4 d w t v+4 w t v-2 d v w^2$$
$$-2 v w^2, -2 v w f+2 v t f+t^3 d+t^3+t^2 w d+t^2 w-t w^2 d-t w^2-w^3 d-w^3+2 w v^2 d+2 w v^2$$

$$-2\,t\,v^2\,d-2\,t\,v^2,\;2+2\,d\,t^2-4\,d\,t\,w+2\,d\,w^2+d\,t^4-2\,d\,w\,t^3+2\,d\,t\,w^3-d\,w^4+2\,f\,v\,w^2$$
$$-4\,f\,w\,t\,v+2\,f\,t^2\,v+4\,t\,g-4\,w\,g+2\,g\,t^3-2\,g\,t^2\,w-2\,g\,t\,w^2+2\,g\,w^3-2\,d^2\}$$

later we will consider the case d=1, f=2 and g=sqrt(3).

```
psirels2:=subs({d=1,f=2,g=sqrt(3)},numer(`minus`(`union`(convert(eva
lm(M1&*psi0-psi0[2]*psi0),set),convert(evalm(M2&*psi0-psi0[3]*psi0),
set),convert(evalm(M3&*psi0-psi0[4]*psi0),set),convert(evalm(M1&*psi
2-psi2[2]*psi2),set),convert(evalm(M2&*psi2-psi2[3]*psi2),set),conve
rt(evalm(M3&*psi2-psi2[4]*psi2),set)),{0})));
```

$$psirels2 := \{2\,n1-2\,n2,\;2\,n3-2\,n5,\;4\,n4-4\,n8,\;2\,n5-2\,n9,\;2\,n7-2\,n11,\;2+2\,n6-2\,n10,$$
$$n1+2\,n2+2\,n3\sqrt{3},\;2\,n2-2\,n4+2,\;-3+n4+2\,n8+2\,n9\sqrt{3},\;n2+2\,n4+2\,n5\sqrt{3}-2,$$
$$-2+n6+2\,n10+n12\sqrt{3}+n13\sqrt{3},\;n3+2\,n5+n6\sqrt{3}+n7\sqrt{3}-\sqrt{3},$$
$$n5+2\,n9+n10\sqrt{3}+n11\sqrt{3}-2\sqrt{3},\;n7+2\,n11+n14\sqrt{3}+n12\sqrt{3}-3\}$$

```
> indets(psirels2);
```

$$\{n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9\}$$

```
> d1basis:=factor(Basis(psirels2,tdeg(n1, n10, n11, n12, n13, n14,
  n2, n3, n4, n5, n6, n7, n8, n9)));
```

$$d1basis := [-3+2\,n9\sqrt{3}+3\,n8,\;-1+n9\sqrt{3}+n7+n6,\;n5-n9,\;-3+2\,n9\sqrt{3}+3\,n4,\;-n9+n3,$$
$$2\,n9\sqrt{3}+3\,n2,\;2\sqrt{3}-3\,n9-2\,n7\sqrt{3}-n14+n13,\;n14+n12+n7\sqrt{3}-\sqrt{3},\;-n7+n11,$$
$$-2+n9\sqrt{3}+n7+n10,\;2\,n9\sqrt{3}+3\,n1]$$

```
> solve(convert(d1basis,set));
```

$$\{n1=-\frac{2\,n9\sqrt{3}}{3},\;n10=2-n9\sqrt{3}-n7,\;n11=n7,\;n12=-n14-n7\sqrt{3}+\sqrt{3},$$
$$n13=-2\sqrt{3}+3\,n9+2\,n7\sqrt{3}+n14,\;n14=n14,\;n2=-\frac{2\,n9\sqrt{3}}{3},\;n3=n9,\;n4=1-\frac{2\,n9\sqrt{3}}{3},$$
$$n5=n9,\;n6=1-n9\sqrt{3}-n7,\;n7=n7,\;n8=1-\frac{2\,n9\sqrt{3}}{3},\;n9=n9\}$$

since these are integers, we see that if d=1 we get the following rules:

```
> d1rules:={n1=0,n9=0,n8=1,n7=1,n10=1,n12=0,n14=0,n13=0,n2=0,n4=1,n5
  =0,n6=0,n3=0,n11=1};
```

$$d1rules := \{n1=0,\;n10=1,\;n11=1,\;n12=0,\;n13=0,\;n14=0,\;n2=0,\;n3=0,\;n4=1,\;n5=0,$$
$$n6=0,\;n7=1,\;n8=1,\;n9=0\}$$

```
> subs(d1rules,[M1,M2,M3]);
```

$$\left[\begin{bmatrix}0&1&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&1&0\end{bmatrix},\begin{bmatrix}0&0&1&0&0\\0&0&1&0&0\\1&1&1&0&0\\0&0&0&1&1\\0&0&0&1&1\end{bmatrix},\begin{bmatrix}0&0&0&0&1\\0&0&0&1&0\\0&0&0&1&1\\1&0&1&0&0\\0&1&1&0&0\end{bmatrix}\right]$$

```
> map(Eigenvalues,%);
```

$$\left[\begin{bmatrix}-1\\-1\\1\\1\\1\end{bmatrix},\begin{bmatrix}-1\\0\\0\\2\\2\end{bmatrix},\begin{bmatrix}0\\\sqrt{3}\\-\sqrt{3}\\I\\-I\end{bmatrix}\right]$$

We combine the nontrivial relations obtained so far.

```
> SandNrels:=`union`(psirels,convert(agb,set),orthrel);
```

$$SandNrels := \{(-w+t)(-3\,v\,w-v\,t+u\,t+u\,w), -(-w+t)(t^3+t^2\,w+2\,t-2\,t\,v^2-t\,w^2+2\,w\,v^2+2\,w-w^3), 2\,d+f^2-2\,g^2, 2+t^2-w^2+2\,u\,v-4\,v^2, 2\,f+t^2\,f-w^2\,f-2\,v\,d-2\,v, 2+d\,t^2-d\,w^2+2\,v\,f+4\,t\,g-2\,g^2, 2\,d+d\,t^2-d\,w^2+2\,v\,f+2\,w\,g+2\,t\,g-2\,g^2, f\,v\,w^2-2\,f\,w\,t\,v+f\,t^2\,v-d\,t^2-t^2+d\,w^2+w^2, 2\,f^2+t^2\,f^2-w^2\,f^2-u\,f\,d-u\,f-d^2-2\,d-1, d\,v\,w^2-2\,d\,w\,t\,v+d\,t^2\,v+f+t^2\,f-w^2\,f-2\,g\,v\,w+2\,g\,v\,t-d\,f, 2\,t\,f-2\,w\,f+f\,t^3-f\,t^2\,w-f\,t\,w^2+f\,w^3+2\,d\,v\,w+2\,v\,w-2\,d\,v\,t-2\,v\,t, -2\,d\,v\,w+2\,d\,v\,t+f\,w^3+f\,t\,w^2-f\,t^2\,w-2\,f\,w\,v^2-f\,t^3+2\,f\,t\,v^2+4\,v\,g-2\,f\,g, 1+d+d\,t^2-d\,w^2+u\,f+g\,w^3+g\,t\,w^2-g\,t^2\,w-2\,g\,w\,v^2-g\,t^3+2\,g\,t\,v^2-f^2, 2\,d\,t-2\,d\,w+t^3\,d-t^2\,w\,d-t\,w^2\,d+w^3\,d-2\,v\,w\,f+2\,v\,t\,f+2\,g\,t^2-2\,g\,w^2+2\,g-2\,d\,g, 2\,t^2\,f-4\,t\,w\,f+2\,w^2\,f+t^4\,f-2\,w\,t^3\,f+2\,t\,w^3\,f-w^4\,f-2\,d\,t^2\,v-2\,t^2\,v+4\,d\,w\,t\,v+4\,w\,t\,v-2\,d\,v\,w^2-2\,v\,w^2, -2\,v\,w\,f+2\,v\,t\,f+t^3\,d+t^3+t^2\,w\,d+t^2\,w-t\,w^2\,d-t\,w^2-w^3\,d-w^3+2\,w\,v^2\,d+2\,w\,v^2-2\,t\,v^2\,d-2\,t\,v^2, 2+2\,d\,t^2-4\,d\,t\,w+2\,d\,w^2+d\,t^4-2\,d\,w\,t^3+2\,d\,t\,w^3-d\,w^4+2\,f\,v\,w^2-4\,f\,w\,t\,v+2\,f\,t^2\,v+4\,t\,g-4\,w\,g+2\,g\,t^3-2\,g\,t^2\,w-2\,g\,t\,w^2+2\,g\,w^3-2\,d^2\}$$

```
> indets(SandNrels);
```

$$\{d, f, g, t, u, v, w\}$$

```
> SandNbasis:=factor(Basis(SandNrels,plex(d, f, g, u, v,t, w)));
```

$$SandNbasis := [-(-w+t)(t^3+t^2\,w+2\,t-2\,t\,v^2-t\,w^2+2\,w\,v^2+2\,w-w^3), -(t^3+t^2\,w+2\,t-2\,t\,v^2-t\,w^2+2\,w\,v^2+2\,w-w^3)(2\,v^2+1+t^2+2\,t\,w-3\,w^2), 2\,t\,v^3-t^3\,v-2\,w\,v^3-t^2\,w\,v+w^2\,t\,v+w^3\,v+u\,t-3\,v\,t+u\,w-5\,v\,w, 2+t^2-w^2+2\,u\,v-4\,v^2, -g\,(t^3+t^2\,w+2\,t-2\,t\,v^2-t\,w^2+2\,w\,v^2+2\,w-w^3), -2-w^2+t^2+2\,g^2-g\,w^3-g\,t^3-6\,t\,g-2\,w\,g-2\,w\,t^3+2\,t\,w^3+2\,w^4-4\,v^2\,w^2+4\,w\,t\,v^2-4\,v^2-2\,t^2\,w^2+g\,t\,w^2+g\,t^2\,w, 2\,f-4\,v-w^2\,f+t^2\,f-2\,g\,v\,t+2\,g\,v\,w, -t\,g-w\,g-w\,t^3+t\,w^3-w^2+w^4+t^2-2\,v^2\,w^2+2\,w\,t\,v^2-2\,v^2-t^2\,w^2+v\,f, 2+u\,f-t\,g-3\,w\,g-2\,w\,t^3+2\,t\,w^3-2\,w^2+2\,w^4+2\,t^2-4\,v^2\,w^2+4\,w\,t\,v^2-4\,v^2-2\,t^2\,w^2, 3\,v\,w$$

$+ v\,t - w^3\,v - w^2\,t\,v + t^2\,w\,v + t^3\,v + 2\,w\,v^3 - 2\,t\,v^3 - 2\,v\,g - v\,g\,w^2 + 2\,g\,v\,t\,w - t^2\,g\,v - w\,f - t\,f$
$+ f\,g, -w^2 + t^2 - 4\,v^2 + 2\,w^4 + 2\,t\,w^3 - 2\,t^2\,w^2 - 4\,v^2\,w^2 - 2\,w\,t^3 - 4\,w\,g + 4\,w\,t\,v^2 - 4\,t\,g - g\,w^3$
$+ g\,t\,w^2 + g\,t^2\,w - g\,t^3 + f^2, -t\,g - 1 + w\,g + d]$

```
>
```

The next step is to introduce the relations involving the roots of unity theta_i. The first goal is to show that th2 satisfies a nontrivial degree 3 (or less) polynomial in Q[d].

```
> psith0:=Vector([1,d*th1,f*th2,g*th3,g*th3]):
> Thetarels:=simplify(subs({},[th1^2*S[2,2]-(sum('M1[k,2]*psith0[k]'
  ,k=1..5)),th1*th2*S[2,3]-(sum('M1[k,3]*psith0[k]',k=1..5)),th1*th3
  *S[2,4]-(sum('M1[k,4]*psith0[k]',k=1..5)),th3*th2*S[4,3]-(sum('M2[
  k,4]*psith0[k]',k=1..5)),th2^2*S[3,3]-(sum('M2[k,3]*psith0[k]',k=1
  ..5)),th3^2*(S[4,4]+S[4,5])-(sum('(M3[k,4]+M3[k,5])*psith0[k]',k=1
  ..5))]));
```

$Thetarels := [th1^2 - 1 - n1\,d\,th1 - n2\,f\,th2 - 2\,n3\,g\,th3, th1\,th2\,f - n2\,d\,th1 - n4\,f\,th2 - 2\,n5\,g\,th3,$
$-th1\,th3\,g - n3\,d\,th1 - n5\,f\,th2 - n6\,g\,th3 - n7\,g\,th3, -n5\,d\,th1 - n9\,f\,th2 - n10\,g\,th3 - n11\,g\,th3,$
$-th2^2\,d - th2^2 - 1 - n4\,d\,th1 - n8\,f\,th2 - 2\,n9\,g\,th3,$
$th3^2\,d - th3^2 - 1 - d\,th1\,n7 - d\,th1\,n6 - f\,th2\,n11 - f\,th2\,n10 - 2\,g\,th3\,n12 - g\,th3\,n13 - g\,th3\,n14]$

```
>
```

we immediately find that th1=1.

```
> factor(Basis(`union`(SandNrels,convert(subs(rules1,Thetarels),set)
  ),tdeg(th1,u, v,t, w,d,th2,th3,f,g)))[7];
```

$$g\,th3\,(th1 - 1)$$

```
> Thetarels1:=factor(numer(subs({th1=1},Thetarels)));
```

$Thetarels1 := [-n1\,d - n2\,f\,th2 - 2\,n3\,g\,th3, f\,th2 - n2\,d - n4\,f\,th2 - 2\,n5\,g\,th3,$
$-g\,th3 - n3\,d - n5\,f\,th2 - n6\,g\,th3 - n7\,g\,th3, -n5\,d - n9\,f\,th2 - n10\,g\,th3 - n11\,g\,th3,$
$-th2^2\,d - th2^2 - 1 - n4\,d - n8\,f\,th2 - 2\,n9\,g\,th3,$
$th3^2\,d - th3^2 - 1 - d\,n7 - d\,n6 - f\,th2\,n11 - f\,th2\,n10 - 2\,g\,th3\,n12 - g\,th3\,n13 - g\,th3\,n14]$

```
> indets(Thetarels1);
```

$$\{d, f, g, n1, n10, n11, n12, n13, n14, n2, n3, n4, n5, n6, n7, n8, n9, th2, th3\}$$

```
> map(degree,Thetarels1,th3);
```

$$[1, 1, 1, 1, 1, 2]$$

```
> solve(Thetarels1[2],th3);
```

$$-\frac{-f\,th2 + n2\,d + n4\,f\,th2}{2\,n5\,g}$$

This is valid unless n5=0, which implies d=1 (which we consider below).

```
> factor(Basis(subs(rules1,`union`(SandNrels,{n5})),plex(u,t,v,w,f,g
  ,d)));
```

$[(d - 1)\,(d + 1), 2\,d + f^2 - 2\,g^2, d\,g^2 + 4\,w\,g - g^2 - 2 + 2\,d, 4\,v - f - d\,f, d\,g + 2\,t - 2\,d\,w - g,$
$u\,(d - 1), 2\,f + 2\,d\,f - f\,g^2 - d\,f\,g^2 - 2\,u + 2\,g^2\,u, 2\,d - g^2 + u\,f - d\,g^2 + 2, -w\,(d\,f + f - u)]$

We substitute back in to eliminate th3 and take at the numerators of the resulting rational functions, which give us our new relations.

```
> Thetarels2:=factor(numer(subs({th3=solve(Thetarels1[2],th3)},Theta
```

```
rels1)));
```

$$Thetarels2 := [-n1\,d\,n5 - n2\,f\,th2\,n5 - n3\,f\,th2 + n3\,n2\,d + n3\,n4\,f\,th2, 0, -f\,th2 + n2\,d + n4\,f\,th2 - 2\,n3\,d\,n5 - 2\,n5^2\,f\,th2 - n6\,f\,th2 + n6\,n2\,d + n6\,n4\,f\,th2 - n7\,f\,th2 + n7\,n2\,d + n7\,n4\,f\,th2, -2\,n5^2\,d - 2\,n9\,f\,th2\,n5 - f\,th2\,n10 + n10\,n2\,d + n10\,n4\,f\,th2 - f\,th2\,n11 + n11\,n2\,d + n11\,n4\,f\,th2, -th2^2\,d\,n5 - th2^2\,n5 - n5 - n4\,d\,n5 - n8\,f\,th2\,n5 - n9\,f\,th2 + n9\,n2\,d + n9\,n4\,f\,th2, 4\,n12\,n5\,g^2\,n4\,f\,th2 + 2\,n13\,n5\,g^2\,n4\,f\,th2 + 2\,n14\,n5\,g^2\,n4\,f\,th2 + 2\,f^2\,th2^2\,n4 - n4^2\,f^2\,th2^2 + d\,f^2\,th2^2 - 4\,f\,th2\,n10\,n5^2\,g^2 - 4\,f\,th2\,n11\,n5^2\,g^2 - 2\,n2\,d\,n4\,f\,th2 + 2\,n2\,d^2\,n4\,f\,th2 - 4\,n12\,n5\,g^2\,f\,th2 + 4\,n12\,n5\,g^2\,n2\,d - 2\,n13\,n5\,g^2\,f\,th2 + 2\,n13\,n5\,g^2\,n2\,d - 2\,n14\,n5\,g^2\,f\,th2 + 2\,n14\,n5\,g^2\,n2\,d - f^2\,th2^2 - n2^2\,d^2 - 4\,n5^2\,g^2 - 4\,d\,n7\,n5^2\,g^2 - 4\,d\,n6\,n5^2\,g^2 + 2\,f\,th2\,n2\,d - 2\,f\,th2\,n2\,d^2 - 2\,d\,f^2\,th2^2\,n4 + d\,n4^2\,f^2\,th2^2 + n2^2\,d^3]$$

```
> Thetarels2red:=factor(Reduce(subs(rules1,Thetarels2),SandNbasis,pl
  ex(d, f, g, u, v,t, w,th2))):

> map(degree,Thetarels2red,th2);
```

$$[-\infty, -\infty, -\infty, -\infty, 2, 2]$$

```
> factor([coeff(Thetarels2red[5],th2,2),coeff(Thetarels2red[5],th2,1
  ),coeff(Thetarels2red[5],th2,0)]);
```

$$[-2\,v\,(-w+t)\,(t\,g+2-w\,g), 4\,t^3\,v + 12\,v\,t + 4\,t^2\,w\,v + 2\,t^2\,g\,v - 8\,t\,v^3 - 4\,w^2\,t\,v + 8\,w\,v^3 - 4\,w^3\,v - 4\,g\,v\,t\,w - 4\,t\,f + 2\,v\,g\,w^2 + 4\,v\,w - 4\,w\,f, -2\,v\,(-w+t)\,(t\,g+2-w\,g)]$$

provided v(t-w) is not zero, th2 satisfies a nonzero degree 2 polynomial (observe that g cannot be rational). The following shows that if v(t-w)=0, then d=1, which we will consider separately.

```
> factor(Basis(`union`(SandNrels,{v*(t-w)}),plex(f,g,u,v,t,w,d)));
```

$$[(d-1)\,(d+1), (-w+t)\,(d+1), -w^2+t^2-d+1, v\,(d-1), v\,(-w+t), w\,(4\,v^2+d+1), u\,(d-1), w\,(u-4\,v), u\,t-4\,v\,w, 2\,u\,v+1-4\,v^2+d, d\,g+2\,t-2\,d\,w-g, w\,(d\,w+g-t), 1-d+t\,g+d\,w^2-t\,w, 1-3\,d+2\,g^2-4\,v^2+4\,d\,w^2-4\,t\,w, f+d\,f-4\,v, v\,(-2\,v+f), u\,f+1-4\,v^2+d, f\,g-2\,v\,g-t\,f-w\,f+4\,v\,w, 1-d-4\,t\,w-4\,v^2+4\,d\,w^2+f^2]$$

```
> ThetaElim:=subs(rules1,`union`(convert(Thetarels,set),{th1-1}));
```

$$ThetaElim := \{th1-1, -(-v\,w+v\,t)\,d\,th1 - \left(\frac{1}{2}w^3 + \frac{1}{2}t\,w^2 - \frac{1}{2}t^2\,w - w\,v^2 - \frac{1}{2}t^3 + t\,v^2\right)f\,th2 - 2\,v\,g\,th3, th1\,th2\,f - (v\,w^2 - 2\,w\,t\,v + t^2\,v)\,d\,th1 - (1+t^2-w^2)\,f\,th2 - 2\,(-v\,w+v\,t)\,g\,th3, th1^2 - 1 - \left(t^2 - 2\,t\,w + w^2 + \frac{1}{2}t^4 - w\,t^3 + t\,w^3 - \frac{1}{2}w^4\right)d\,th1 - (v\,w^2 - 2\,w\,t\,v + t^2\,v)\,f\,th2 - 2\left(t - w + \frac{1}{2}t^3 - \frac{1}{2}t^2\,w - \frac{1}{2}t\,w^2 + \frac{1}{2}w^3\right)g\,th3, -th1\,th3\,g - \left(t - w + \frac{1}{2}t^3 - \frac{1}{2}t^2\,w - \frac{1}{2}t\,w^2 + \frac{1}{2}w^3\right)d\,th1 - (-v\,w+v\,t)\,f\,th2 - \left(\frac{t^2}{2} - \frac{w^2}{2}\right)g\,th3$$

$$-\left(1+\frac{t^2}{2}-\frac{w^2}{2}\right) g\, th3,\ -th2^2\, d - th2^2 - 1 - (1+t^2-w^2)\, d\, th1 - u\, f\, th2$$

$$-2\left(\frac{1}{2}w^3+\frac{1}{2}t\,w^2-\frac{1}{2}t^2\,w - w\,v^2 - \frac{1}{2}t^3 + t\,v^2\right) g\, th3,$$

$$th3^2\, d - th3^2 - 1 - d\, th1\left(1+\frac{t^2}{2}-\frac{w^2}{2}\right) - d\, th1\left(\frac{t^2}{2}-\frac{w^2}{2}\right) - 2\, f\, th2\, v - 3\, g\, th3\, t - g\, th3\, w\}$$

```
> indets(ThetaElim);
```

$$\{d, f, g, t, th1, th2, th3, u, v, w\}$$

```
> indets(allNrels2);
```

$$\{m, n1, n10, n12, n2, n3, n4, n5, n6, n7, n9, t, u, v, w\}$$

first suppose d=1. This forces th3=0.

```
> factor(Basis(subs(‘union‘({d=1,f=2,g=sqrt(3)},d1rules),convert(The
  tarels,set)),tdeg(th1,th3,th2)));
```

$$[th2,\ th3,\ th1+1]$$

```
> with(numtheory):
```

Since [Q[d,f,g]:Q]=2, th2 satisfies a cyclotomic polynomial of degree 1,2 or 4.

```
> invphi(4); invphi(2);
```

$$[5, 8, 10, 12]$$

$$[3, 4, 6]$$

```
> bases1:=i->factor(Basis(‘union‘(SandNrels,ThetaElim,{cyclotomic(i,
  th2)}),plex(d,f,g,th1,th3,th2,u,v,t,w)))[1];
```

$$bases1 := i \rightarrow \mathrm{factor}(Groebner\text{:-}Basis(\text{‘union‘}(SandNrels,\ ThetaElim,\ \{numtheory\text{:-}cyclotomic(i,\ th2)\}),\ \mathrm{plex}(d, f, g, th1, th3, th2, u, v, t, w)))_1$$

```
> map(bases1,invphi(4));
```

$$[(22201 - 19008\, w^6 + 20736\, w^8 - 201456\, w^4 - 65532\, w^2)\,(t^2 - 2\,t\,w + 4 + w^2),$$

$$(-36481 - 89856\, w^6 + 10368\, w^8 + 159984\, w^4 + 19704\, w^2)\,(t^2 - 2\,t\,w + 4 + w^2),$$

$$(-58081 - 302400\, w^6 + 103680\, w^8 + 187920\, w^4 + 75660\, w^2)\,(t^2 - 2\,t\,w + 4 + w^2),$$

$$(6\,w^2 + 18\,w + 13)\,(6\,w^2 - 18\,w + 13)\,(288\,w^4 + 24\,w^2 - 121)\,(t^2 - 2\,t\,w + 4 + w^2)]$$

```
> map(bases1,invphi(2));
```

$$[(9\,w^2+4)\,(t^2 - 2\,t\,w + 4 + w^2),\ (-1 + 60\,w^2 + 72\,w^4)\,(t^2 - 2\,t\,w + 4 + w^2),\ t^2 - 2\,t\,w + 4 + w^2]$$

clearly have no solutions.

```
> expand((t-w)^2+4);
```

$$t^2 - 2\,t\,w + 4 + w^2$$

finally we see that if th2^2=1, we get d=1.

```
> factor(Basis(‘union‘(SandNrels,ThetaElim,{th2^2-1}),plex(f,g,th1,t
  h3,th2,u,v,t,w,d)));
```

$$[(d-1)^2\,(d+1)^2,\ -(d-1)\,(d+1)\,(25\,d - 25 - 128\,w^2),\ -(d-1)\,(d+1)\,(14\,w\,d - 19\,w - 15\,t),$$

$$-(d-1)\,(d^2 + 2\,d - 3 + 2\,t\,w - w^2 - t^2),$$

$4-6\,d+2\,t\,w^3+6\,t^2-w^4+2\,d^3-4\,t\,w-2\,w\,t^3-2\,w^2+t^4$, $11+29\,d+64\,v^2+5\,d^2-16\,t\,w^3$
$-16\,t^2-16\,w^2\,d+16\,w^4-13\,d^3+16\,d\,w\,t+16\,t\,w+16\,w\,t^3-16\,t^2\,w^2$,
$-29\,v+6\,u+6\,d\,u-13\,v\,d+12\,w\,t\,v-6\,t^2\,v-7\,d^2\,v+v\,d^3-6\,v\,w^2$, $-219\,v+64\,u+64\,u\,w^2$
$+16\,w^2\,v\,d-53\,v\,d+80\,w\,t\,v-32\,t^2\,v+64\,v\,w^3\,t-32\,t^2\,w^2\,v-16\,d\,w\,t\,v-32\,v\,w^4-5\,d^2\,v$
$+21\,v\,d^3-304\,v\,w^2$, $-105\,v\,t+30\,u\,t-15\,w^3\,v-142\,v\,w+15\,v\,w\,t^2+15\,t\,w^2\,v-15\,v\,t^3$
$+7\,w\,d^2\,v+32\,v\,w\,d-17\,v\,d^3\,w-15\,d\,v\,t+30\,u\,w$, $43+32\,u\,v+29\,d+5\,d^2-16\,t\,w^3-16\,w^2\,d$
$+16\,w^4-13\,d^3+16\,d\,w\,t+16\,t\,w+16\,w\,t^3-16\,t^2\,w^2-16\,w^2$, $(d+1)\,(d^2-2\,d-1+2\,th2)$,
$(t^2-2\,t\,w+4+w^2)\,(th2-1)$, $(th2-1)\,(th2+1)$, $(d-1)\,(3\,d^2+2\,d-5+4\,th3)$, $28\,v-6\,u-4\,v\,d$
$-3\,th3\,v\,t^2+6\,th3\,v\,t\,w-6\,w\,t\,v+3\,t^2\,v+6\,th3\,u-24\,th3\,v-3\,th3\,v\,w^2-4\,d^2\,v+4\,v\,d^3+3\,v\,w^2$
, $-d-d^2-2\,th2-2\,th3+2\,th2\,th3+3+d^3$, $th1-1$, $(d-1)\,(7\,d^2\,w-10\,w\,d+30\,g-15\,t-2\,w)$,
$-3-3\,d+16\,g\,w+4\,w^2-4\,w^2\,d+3\,d^2+3\,d^3+4\,d\,w\,t-4\,t\,w$,
$13-19\,d+4\,w^2+16\,t\,g-4\,w^2\,d+3\,d^2+3\,d^3+4\,d\,w\,t-4\,t\,w$, $9\,v\,t-6\,w^3\,v-52\,v\,w-48\,v\,g$
$-6\,v\,w\,t^2+12\,t\,w^2\,v+12\,g\,u-5\,w\,d^2\,v+6\,v\,w\,d+3\,v\,d^3\,w-9\,d\,v\,t+12\,u\,w$,
$-10\,g+5\,t-w-4\,d^2\,w-4\,w\,d+4\,d^3\,w+10\,th2\,g-5\,th2\,t+5\,th2\,w$,
$13\,w-13\,w\,d-5\,t-8\,d^2\,w+5\,t\,d+8\,d^3\,w+20\,g\,th3$, $1-2\,d+d^2+4\,g^2$,
$-11\,v+5\,v\,d-2\,v\,w^2+4\,w\,t\,v+3\,d^2\,v-2\,t^2\,v-5\,v\,d^3+4\,f$]

>