SOLUTIONS

MATH 151, Spring 2021
COMMON EXAM 1 - ONLINE EXAM VERSION A

The 5 workout problems make up 43 points of the exam and the 19 multiple choice problems make up 57 points (3 points each), for a total of 100 points. **No calculator is allowed!**

PART I: WORK OUT PROBLEMS

Directions: Present each of your solutions on an empty sheet/side of paper. *Show all of your work* neatly and concisely and *box your final answer*. You will be graded not merely on the final answer, but also the quality and correctness of the work leading up to it.

1. (8 pts) Use the **definition** of the derivative to find $f'(x)$ for $f(x) = \sqrt{8-x}$. *No shortcuts are allowed!*

2. (8 pts) Consider the function $g(x) = \begin{cases} 3x^2 + A, & \text{if } x < -2, \\ Bx - 7, & \text{if } x = -2, \\ Ax + 3, & \text{if } x > -2. \end{cases}$

   Find the values of A and B that will make $g(x)$ continuous. If no such values exist, then explain why.

3. (7 pts) A pilot steers a plane in the direction 210° counterclockwise from the positive x-axis at a speed of 400 mph. The wind is blowing in a direction 60° counterclockwise from the positive x-axis at a speed of 20 mph. Find the true (resultant) velocity **vector** of the plane. (Your answer should be a vector.)

4. (14 pts) Evaluate these limits. Do not use the L'Hôpital method.

   (a) $\displaystyle\lim_{x\to 7^+} \frac{x^2 - 4x - 21}{|7-x|}$ (b) $\displaystyle\lim_{x\to 3} \frac{\frac{1}{x} - \frac{1}{3}}{3x - x^2}$ (c) $\displaystyle\lim_{x\to -4} \frac{x+4}{\sqrt{x^2+9} - 5}$

5. (6 pts) Consider the function $f(x) = \begin{cases} 16 - x^2, & \text{if } x \leq -1, \\ 8 - 5x, & \text{if } -1 < x \leq 3, \\ x^3 + 1, & \text{if } x > 3. \end{cases}$

   (a) Evaluate $\displaystyle\lim_{x\to 3} f(x)$, or explain why it does not exist.

   (b) Evaluate $\displaystyle\lim_{x\to -1^+} f(x)$, or explain why it does not exist.

Multiple choice problems begin on the next page...

1. (8 pts) Use the **definition** of the derivative to find $f'(x)$ for $f(x) = \sqrt{8-x}$. *No shortcuts are allowed!*

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{8-(x+h)}-\sqrt{8-x}}{h} = \frac{\sqrt{8-x-h}-\sqrt{8-x}}{h} \cdot \frac{\sqrt{8-x-h}+\sqrt{8-x}}{\sqrt{8-x-h}+\sqrt{8-x}}$$

$$= \frac{8-x-h-(8-x)}{h\left(\sqrt{8-x-h}+\sqrt{8-x}\right)} = \frac{8-x-h-8+x}{h\left(\sqrt{8-x-h}+\sqrt{8-x}\right)} = \frac{-h}{h\left(\sqrt{8-x-h}+\sqrt{8-x}\right)} = \frac{-1}{\sqrt{8-x-h}+\sqrt{8-x}}$$

$$\xrightarrow[\text{as } h \to 0]{} \frac{-1}{\sqrt{8-x-0}+\sqrt{8-x}} = \boxed{\frac{-1}{2\sqrt{8-x}}}$$

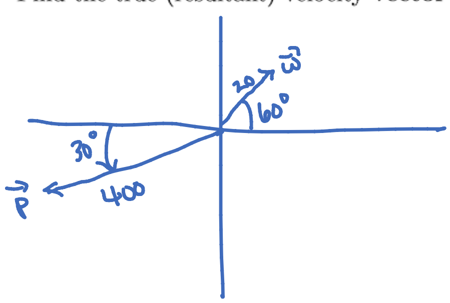
2. (8 pts) Consider the function $g(x) = \begin{cases} 3x^2 + A, & \text{if } x < -2, \\ Bx - 7, & \text{if } x = -2, \\ Ax + 3, & \text{if } x > -2. \end{cases}$

Find the values of A and B that will make $g(x)$ continuous. If no such values exist, then explain why.

Need $\lim_{x \to -2^+} g(x) = \lim_{x \to -2^-} g(x) \Rightarrow A(-2)+3 = 3(-2)^2 + A \Rightarrow -9 = 3A \Rightarrow \boxed{A = -3}$

Need $f(-2) = \lim_{x \to -2} g(x) \Rightarrow B(-2) - 7 = \underbrace{(-3)}_{A}(-2) + 3 \Rightarrow -2B = 16 \Rightarrow \boxed{B = -8}$

3. (7 pts) A pilot steers a plane in the direction 210° counterclockwise from the positive x-axis at a speed of 400 mph. The wind is blowing in a direction 60° counterclockwise from the positive x-axis at a speed of 20 mph. Find the true (resultant) velocity **vector** of the plane. (Your answer should be a vector.)



plane vec $= \vec{p} = \langle -400\cos 30°, -400 \sin 30° \rangle$ mph

$= \langle -200\sqrt{3}, -200 \rangle$ mph

wind vec $= \vec{w} = \langle 20 \cos 60°, 20 \sin 60° \rangle$ mph

$= \langle 10, 10\sqrt{3} \rangle$ mph

$\boxed{\text{resultant} = \langle -200\sqrt{3} + 10, -200 + 10\sqrt{3} \rangle \text{ mph}}$

4. (14 pts) Evaluate these limits. Do not use the L'Hôpital method.

(a) $\lim\limits_{x\to 7+} \dfrac{x^2-4x-21}{|7-x|}$ (b) $\lim\limits_{x\to 3} \dfrac{\frac{1}{x}-\frac{1}{3}}{3x-x^2}$ (c) $\lim\limits_{x\to -4} \dfrac{x+4}{\sqrt{x^2+9}-5}$

(a) $\lim\limits_{x\to 7^+} \dfrac{x^2-4x-21}{|7-x|} = \lim\limits_{x\to 7^+} \dfrac{(x-7)(x+3)}{\underbrace{|7-x|}} = \lim\limits_{x\to 7^+} \dfrac{(x-7)(x+3)}{(x-7)} = \lim\limits_{x\to 7^+} (x+3) = (7+3) = \boxed{10}$

$x>7 \Rightarrow 7-x<0 \Rightarrow |7-x| = -(7-x) = x-7$

(b) $\lim\limits_{x\to 3} \dfrac{1/x - 1/3}{3x-x^2} = \lim\limits_{x\to 3} \dfrac{\frac{3-x}{3x}}{x(3-x)} = \lim\limits_{x\to -3} \dfrac{3-x}{3x^2(3-x)} = \lim\limits_{x\to -3} \dfrac{1}{3x^2} = \boxed{\dfrac{1}{27}}$

(c) $\lim\limits_{x\to -4} \dfrac{x+4}{\sqrt{x^2+9}-5} \cdot \dfrac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5} = \lim\limits_{x\to -4} \dfrac{(x+4)(\sqrt{x^2+9}+5)}{x^2+9-25} = \lim\limits_{x\to -4} \dfrac{(x+4)(\sqrt{x^2+9}+5)}{x^2-16}$

$= \lim\limits_{x\to -4} \dfrac{(x+4)(\sqrt{x^2+9}+5)}{(x+4)(x-4)} = \lim\limits_{x\to -4} \dfrac{\sqrt{x^2+9}+5}{x-4} = \dfrac{\sqrt{16+9}+5}{-4-4} = \dfrac{10}{-8} = \boxed{-\dfrac{5}{4}}$

5. (6 pts) Consider the function $f(x) = \begin{cases} 16-x^2, & \text{if } x \le -1, \\ 8-5x, & \text{if } -1 < x \le 3, \\ x^3+1, & \text{if } x > 3. \end{cases}$

(a) Evaluate $\lim\limits_{x\to 3} f(x)$, or explain why it does not exist.

(b) Evaluate $\lim\limits_{x\to -1+} f(x)$, or explain why it does not exist.

(a) $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} (8-5x) = -7$

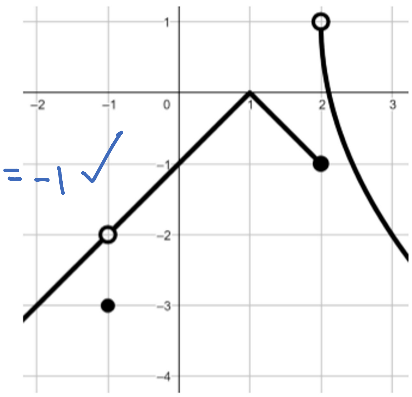
$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} x^3+1 = 28$

NOT equal $\Rightarrow$ $\boxed{\text{limit DNE}}$

(b) $\lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (8-5x) = \boxed{13}$

PART II: Multiple Choice. 3 points each

6. The following is the graph of a function $y = f(x)$. Which of the following statements concerning the graph is **TRUE**?



(a) $f(2) = 1$ $f(2) = -1$

(b) $f(x)$ is continuous at $x = -1$ (hole)

(c) $f(x)$ is continuous from the left at $x = 2$ ← correct $\lim_{x \to 2^-} f(x) = -1$ ✓

(d) $\lim_{x \to 2^+} f(x) = -1$, no equals 1

(e) $\lim_{x \to -1} f(x)$ does not exist $\lim_{x \to -1} f(x) = -2$

7. The displacement (in feet) of a particle moving in a straight line is given by $s(t) = t^2 + 5t - 6$, where t is measured in seconds. Find the average velocity over the interval $[1, 4]$.

(a) $\frac{15}{2}$ ft/s

(b) 30 ft/s

(c) $\frac{10}{3}$ ft/s

(d) 0 ft/s

(e) 10 ft/s ← correct

$$\text{avg. vel} = \frac{\Delta s}{\Delta t} = \frac{s(4) - s(1)}{4 - 1}$$

$$= \frac{(16 + 20 - 6) - (1 + 5 - 6)}{3} = \frac{30}{3} = 10 \text{ ft/s}$$

8. Given $\mathbf{a} = \langle 2, -1 \rangle$ and $\mathbf{b} = \langle 1, -3 \rangle$, find the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$.

(a) $\frac{5}{\sqrt{5}}$ ← correct

(b) $\frac{1}{2}\mathbf{i} - \frac{3}{2}\mathbf{j}$

(c) 1

(d) $2\mathbf{i} - \mathbf{j}$

(e) $\frac{5}{\sqrt{10}}$

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{b} \cdot \vec{a}}{|\vec{a}|} = \frac{2 + 3}{\sqrt{2^2 + (-1)^2}} = \boxed{\frac{5}{\sqrt{5}}}$$

9. Which of the following are parametric equations of a line passing through the point $(3, -7)$ and perpendicular to the vector $\langle 6, -5 \rangle$?

(a) $x = -7 - 6t, y = 3 + 5t$

(b) $x = 3 + 5t, y = -7 + 6t$ ← correct

(c) $x = 3 - 6t, y = -7 + 5t$

(d) $x = -7 + 5t, y = 3 + 6t$

(e) $x = 3 - 5t, y = -7 + 6t$

parallel vec is $\vec{v} = \langle 5, 6 \rangle$

pt: $(3, -7) \Rightarrow \vec{r_0} = \langle 3, -7 \rangle$

$\Rightarrow \vec{r}(t) = \vec{r_0} + t\vec{v} = \langle 3 + 5t, -7 + 6t \rangle$

$\Rightarrow \boxed{x = 3 + 5t, \ y = -7 + 6t}$

10. Evaluate $\lim_{x \to 3} \frac{2x^2+6x}{x^2-2x-15}$.

(a) 3
(b) ∞
(c) -3 ← correct
(d) $-\infty$
(e) $-\frac{3}{2}$

$$\frac{2x^2+6x}{x^2-2x-15} = \frac{2x\cancel{(x+3)}}{\cancel{(x+3)}(x-5)} \xrightarrow{\text{as } x \to 3} \frac{2\cdot 3}{3-5} = \boxed{-3}$$

11. Evaluate $\lim_{x \to -\infty} \frac{8x+3}{\sqrt{5x+16x^2}}$.

(a) -2 ← correct
(b) $\frac{8}{\sqrt{5}}$
(c) $\frac{1}{2}$
(d) $-\frac{1}{2}$
(e) 2

$$\lim_{x \to -\infty} \frac{8x+3}{\sqrt{5x+16x^2}} \cdot \frac{1/x}{1/x} = \lim_{x \to -\infty} \frac{8+3/x}{-\sqrt{5/x+16}} = \frac{8+0}{-\sqrt{0+16}} = \frac{8}{-4} = \boxed{-2}$$

$= -\sqrt{1/x^2}$ since $x<0$

12. Find the point of intersection of the following two lines, if it exists.

$$L_1 : \mathbf{r}(t) = \langle 7-3t, 4+t \rangle$$
$$L_2 : \mathbf{r}(s) = \langle 1+s, 2+s \rangle$$

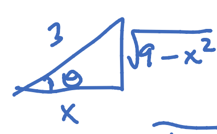
(a) the lines do not interest
(b) $(1,3)$
(c) $(2,5)$
(d) $(3,4)$
(e) $(4,5)$ ← correct

$7-3t = 1+s \Rightarrow s = 6-3t \Rightarrow 4+t = 2+(6-3t)$
$4+t = 2+s \qquad \Rightarrow 4+t = 8-3t$
$\Rightarrow 4t = 4 \Rightarrow t = 1$
$\Rightarrow$ pt: $(7-3\cdot 1, 4+1) = \boxed{(4,5)}$

13. Simplify $\sin\left(\arccos\left(\frac{x}{3}\right)\right)$, $-3 < x < 3$, to an algebraic expression.

(a) $\frac{\sqrt{9-x^2}}{3}$ ← correct
(b) $\frac{x}{\sqrt{9-x^2}}$
(c) $\frac{\sqrt{9+x^2}}{x}$
(d) $\frac{3}{\sqrt{9-x^2}}$
(e) $\frac{\sqrt{9-x^2}}{x}$

Set $\theta = \arccos\left(\frac{x}{3}\right) \Rightarrow \cos\theta = \frac{x}{3}$



$\Rightarrow \sin\left(\arccos\left(\frac{x}{3}\right)\right) = \sin\theta = \boxed{\frac{\sqrt{9-x^2}}{3}}$

14. Evaluate $\lim_{x \to -\infty} \arctan\left(\frac{6x + x^3}{2x - 5x^2}\right)$.

(a) ∞

(b) 0

(c) $-\infty$

(d) $\frac{\pi}{2}$ ← correct

(e) $-\frac{\pi}{2}$

$\frac{6x+x^3}{2x-5x^2} \longrightarrow \infty$ as $x \to -\infty$

$\Rightarrow \arctan\left(\frac{6x+x^3}{2x-5x^2}\right) \longrightarrow \boxed{\frac{\pi}{2}}$ as $x \to -\infty$

15. Which of the following statements is true regarding the equation $4 = x^5 + 2x$?

(a) A solution exists on the interval $(1, 2)$ by the Squeeze Theorem

(b) A solution exists on the interval $(1, 2)$ by the Intermediate Value Theorem ← correct

(c) A solution exists on the interval $(0, 1)$ by the Squeeze Theorem

(d) A solution exists on the interval $(0, 1)$ by the Intermediate Value Theorem

(e) the equation has no real number solutions

Set $f(x) = x^5 + 2x - 4$

$f(0) = -4 < 0$

$f(1) = 1 + 2 - 4 = -1 < 0$

$f(2) = 32 + 4 - 4 = 32 > 0$

$\Rightarrow$ $\boxed{\text{sol}^{n} \text{ on } (1,2) \text{ by IVT}}$

16. Find the cosine of the angle between the vectors $\langle -4, 2\rangle$ and $\langle 1, 5\rangle$.

(a) $\frac{6}{\sqrt{20}\sqrt{26}}$ ← correct

(b) $-\frac{40}{\sqrt{20}}$

(c) $\frac{6}{\sqrt{26}}$

(d) $-\frac{40}{\sqrt{20}\sqrt{26}}$

(e) $\frac{6}{\sqrt{20}}$

$\vec{a} = \langle -4, 2\rangle, \quad \vec{b} = \langle 1, 5\rangle$

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{(-4)(1) + (2)(5)}{\sqrt{(-4)^2+2^2}\,\sqrt{(1)^2+5^2}} = \boxed{\frac{6}{\sqrt{20}\sqrt{26}}}$

17. A sled is pulled along a level path by a rope. A 30-lb force acting an angle 60° above the horizontal moves the sled 6 ft. Find the work done by the force.

(a) $90\sqrt{3}$ J

(b) 180 J

(c) $90\sqrt{2}$ J

(d) 45 J

(e) 90 J ← correct

$W = (30)(6)\cos 60^\circ = \boxed{90 \text{ J}}$

18. Evaluate $\lim_{x\to\infty} \frac{7e^{6x} - 5e^{-3x}}{4e^{6x} + 2e^{-3x}}$.

(a) $-\frac{5}{2}$

(b) 0

(c) $\frac{7}{4}$ ← correct

(d) ∞

(e) $-\infty$

$\lim_{x\to\infty} \frac{7\overbrace{e^{6x}}^{\to\infty} - 5\overbrace{e^{-3x}}^{\to 0}}{4e^{6x} + 2e^{-3x}} \cdot \frac{e^{-6x}}{e^{-6x}} = \lim_{x\to\infty} \frac{7 - 5e^{-9x}}{4 + 2e^{-9x}} = \frac{7 - 0}{4 + 0} = \boxed{\frac{7}{4}}$

19. Find the distance between the point $(-2,5)$ to the line $y = \frac{1}{3}x + 1$.

(a) $\frac{17}{\sqrt{29}}$

(b) $\frac{14}{\sqrt{10}}$ ← correct

(c) $\frac{2}{\sqrt{10}}$

(d) $\frac{14}{\sqrt{20}}$

(e) $\frac{2}{\sqrt{20}}$

$(-2,5)$ $(0,1)$ $m = \frac{1}{3}$

$\vec{v} = \langle -2-0, 5-1 \rangle$ $\Rightarrow$ vec paralel is $\langle 3,1 \rangle$

$= \langle -2, 4 \rangle$ $\Rightarrow$ $\vec{n}$ = vec $\perp$ = $\langle -1, 3 \rangle$

$\Rightarrow$ distance $= \left| \text{comp}_{\vec{n}} \vec{v} \right| = \left| \frac{\vec{v} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{2+12}{\sqrt{10}} \right| = \boxed{\frac{14}{\sqrt{10}}}$

20. Find all vertical asymptotes of the function $f(x) = \frac{x^2 - 9}{x^3 + x^2 - 12x}$

(a) $x = 0$, $x = -4$, and $x = 3$

(b) $x = 0$ and $x = -4$ ← correct

(c) $x = 0$ and $x = 3$

(d) $x = -4$ and $x = 3$

(e) $x = -4$ only

$f(x) = \frac{(x+3)(x-3)}{x(x+4)(x-3)}$ $\Rightarrow$ $x = 0, -4$ are VA's

21. Find a vector of length 4 in the same direction as the vector from the point $(3,-5)$ and $(-2,7)$.

(a) $\langle -20, 48 \rangle$

(b) $\left\langle -\frac{20}{\sqrt{13}}, \frac{48}{\sqrt{13}} \right\rangle$

(c) $\left\langle -\frac{5}{4}, 3 \right\rangle$

(d) $\left\langle -\frac{20}{13}, \frac{48}{13} \right\rangle$ ← correct

(e) $\left\langle \frac{20}{13}, -\frac{48}{13} \right\rangle$

$\vec{a} = \langle -2-3, 7-(-5) \rangle = \langle -5, 12 \rangle$

$\Rightarrow$ unit vec $= \frac{\vec{a}}{|\vec{a}|} = \left\langle \frac{-5}{13}, \frac{12}{13} \right\rangle$

$\Rightarrow$ lenth 4 vec is $\boxed{\left\langle \frac{-20}{13}, \frac{48}{13} \right\rangle}$

22. Evaluate $\lim_{x \to -5^+} \frac{2-x}{x^2 + 4x - 5}$.

(a) $-\infty$ ← correct

(b) -1

(c) 0

(d) $-\frac{2}{5}$

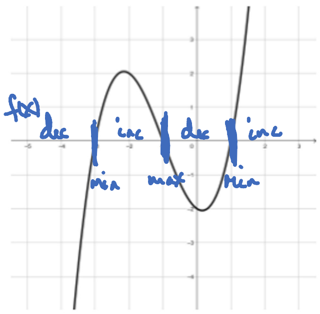
(e) ∞

$\frac{2-x}{x^2+4x-5} = \frac{2-x}{(x+5)(x-1)}$ $\Rightarrow$ $x = -5$ is a VA so the limit is $\pm\infty$

$x \to -5^+$ means x is close to -5 & $x > -5$

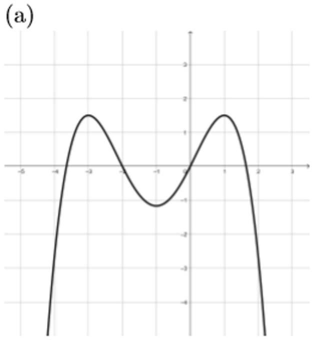
$\Rightarrow$ signs of each factor: $\frac{\oplus}{\oplus \ominus} = \ominus$ $\Rightarrow$ $\boxed{-\infty}$

23. The following is the graph of graph of $f'(x)$.

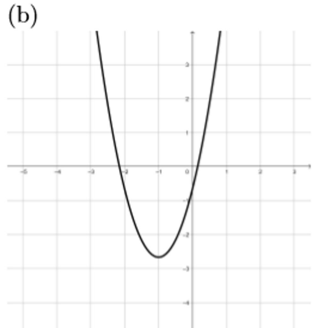


Which of the following is the graph of $f(x)$?

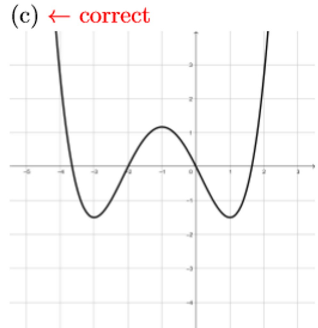
(a)



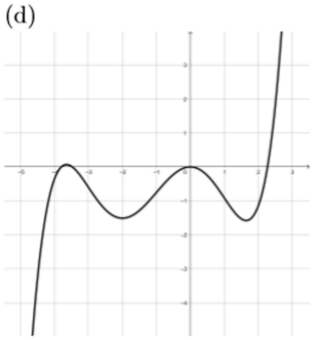
(b)



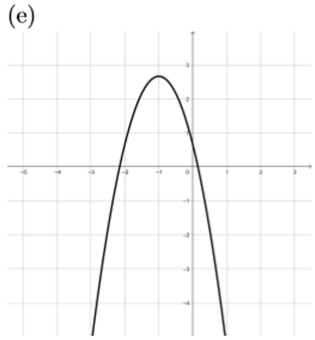
(c) ← correct



(d)



(e)



24. Given $f(x) = x^3 - 3x + 1$ and $f'(x) = 3x^2 - 3$, find the equation of the tangent line to $f(x)$ at $x = -2$.

(a) $y = 3x - 1$

(b) $y = 3x + 17$

(c) $y = 9x - 1$

(d) $y = 3x + 5$

(e) $y = 9x + 17$ ← correct

slope: $f'(-2) = 3(-2)^2 - 3 = 12 - 3 = 9$

point: $f(-2) = (-2)^3 - 3(-2) + 1 = -8 + 6 + 1 = -1 \Rightarrow (-2, -1)$

$\Rightarrow$ tangent line is $y = 9(x - (-2)) + (-1) = 9x + 18 - 1 = 9x + 17$

$\Rightarrow \boxed{y = 9x + 17}$