MATH 151, SPRING 2021
COMMON EXAM 3 - ONLINE EXAM VERSION A

The work out problems make up 36 points of the exam, while the multiple choice problems make up 64 points (4 points each) for a total of 100 points. **No calculator is allowed!**

PART I: WORK OUT PROBLEMS

Directions: Present each of your solutions on an empty sheet/side of paper. *Show all of your work* neatly and concisely and *box your final answer*. You will be graded not merely on the final answer, but also the quality and correctness of the work leading up to it.

1. (10 pts) A rectangular box with an open top is to have a volume of 90 cubic meters. The length of the box is triple its height. Material for the base costs 2 dollars per square meter, and the material for the sides cost 3 dollars per square meter. Determine the height of the box that minimizes the cost of the container. Justify that your answer gives a minimum using calculus.

   Answer: $\sqrt[3]{10}\,\mathrm{m}$

2. (8 pts) Calculate $\lim_{x\to 0^+} [1+f(x)]^{-5/x}$ if $f(x)$ has a continuous first derivative and satisfies $f(0)=0$ and $f'(0)=2$.

   Answer: e^{-10}

3. (9 pts) A particle is moving at a speed of 32 meters per second before it begins slowing down. The particle decelerates at a non-constant rate of $\boldsymbol{12t^2}$ meters per second squared, where t is the time from when it begins slowing down. How far does the particle travel before coming to a stop?

   Answer: 48 meters

4. (9 pts) Consider a function $f(x)$ that has a vertical asymptote at $x=3$, and for $x\neq 3$ has derivatives:

$$f'(x)=\frac{(x+11)(x-1)^2}{(x+3)^3} \qquad \text{and} \qquad f''(x)=\frac{96(x-1)}{(x+3)^4}.$$

   For each of the following, make sure to justify your answers.

   (a) Determine the interval(s) where f is increasing and decreasing.

   (b) Determine the x-values(s) where any local extrema occur, or argue that there are none. Also state which type of extrema occurs.

   (c) Determine the interval(s) where f is concave up and concave down.

   (d) Determine the x-coordinate(s) of any inflection points, or argue that there are none.

   Answer:

   (a) increasing on $(-\infty,-11),(-3,1),(1,\infty)$ (or $(-\infty,-11),(-3,\infty)$); decreasing on $(-11,-3)$

   (b) at $x=-11$ because f goes from increasing/decreasing or decreasing/increasing

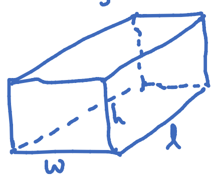
   (c) concave up on $(1,\infty)$; concave down on $(-\infty,-3),(-3,1)$

   (d) only at $x=1$ since concavity changes there

Multiple choice problems begin on the next page...

1. (10 pts) A rectangular box with an open top is to have a volume of 90 cubic meters. The length of the box is triple its height. Material for the base costs 2 dollars per square meter, and the material for the sides cost 3 dollars per square meter. Determine the height of the box that minimizes the cost of the container. Justify that your answer gives a minimum using calculus.

l = length, w = width, h = height, V = volume, C = cost



$$C = 2[wl] + 3[2(lh) + 2(wh)]$$

$$l = 3h \quad \& \quad 90 = V = l \cdot w \cdot h = 3h \cdot w \cdot h = 3wh^2 \Rightarrow w = 30/h^2$$

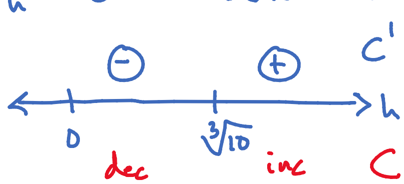
$$\Rightarrow C = 2\left[(30/h^2)(3h)\right] + 3\left[2(3h)(h) + 2(30/h^2)(h)\right]$$

$$= 180/h + 18h^2 + 180/h$$

$$\Rightarrow C = 18h^2 + 360/h \quad \text{with } h > 0$$

$$C' = 36h - 360/h^2 = 0 \Rightarrow 36h^3 - 360 = 0 \Rightarrow h^3 = 10 \Rightarrow h = \sqrt[3]{10}$$

Draw a # line:



(using test pts $h = 1$ & $h = 10000000$)

Since C decreases on $(0, \sqrt[3]{10})$ and increases on $(\sqrt[3]{10}, \infty)$, C has a global minimum at $h = \sqrt[3]{10}$ on its domain $h > 0$.

$\Rightarrow$ $\boxed{h = \sqrt[3]{10} \text{ m gives the minimum cost}}$

2. (8 pts) Calculate $\lim_{x\to 0^+} [1+f(x)]^{-5/x}$ if $f(x)$ has a continuous first derivative and satisfies $f(0)=0$ and $f'(0)=2$.

$$\lim_{x\to 0^+} [1+f(x)]^{-5/x} = \lim_{x\to 0^+} e^{\ln\left([1+f(x)]^{-5/x}\right)} = \lim_{x\to 0^+} e^{-\frac{5}{x}\ln[1+f(x)]} = \boxed{e^{-10}}$$

Do first

$$\lim_{x\to 0^+} \overbrace{-\frac{5}{x}\ln[1+f(x)]}^{\substack{\infty\cdot 0 \text{ form} \\ \text{since } f(0)=0}} = \lim_{x\to 0^+} \frac{\overbrace{-5\ln[1+f(x)]}^{0/0 \text{ form}}}{x}$$

$$\overset{\text{L.H.}}{=} \lim_{x\to 0^+} \frac{-5\cdot\frac{f'(x)}{1+f(x)}}{1}$$

$$= -5\cdot\frac{f'(0)}{1+f(0)} = -5\cdot\frac{2}{1+0} = -10$$

3. (9 pts) A particle is moving at a speed of 32 meters per second before it begins slowing down. The particle decelerates at a <u>non-constant</u> rate of $\mathbf{12t^2}$ meters per second squared, where t is the time from when it begins slowing down. How far does the particle travel before coming to a stop?

$a(t) = -12t^2$, where it's negative since it's decelerating

antideriv. $\Rightarrow v(t) = -4t^3 + C$, where $32 = v(0) = 0 + C \Rightarrow C = 32 \Rightarrow v(t) = -4t^3 + 32$

Setting this equal to zero gives when the particle stops: $v(t) = 0$.

$\Rightarrow -4t^3 + 32 = 0 \Rightarrow t^3 = \frac{32}{4} = 8 \Rightarrow t = 2$

Get the displacement function:

antideriv of $v(t)$ $\Rightarrow s(t) = -t^4 + 32t + D$, where $0 = s(0) = 0 + 0 + D \Rightarrow D = 0 \Rightarrow s(t) = -t^4 + 32t$

$\Rightarrow s(2) = -2^4 + 32(2) = -16 + 64 = 48$

$\Rightarrow$ $\boxed{\text{The particle stops after traveling 48 m}}$

4. (9 pts) Consider a function $f(x)$ that has a vertical asymptote at $x = 3$, and for $x \neq 3$ has derivatives:

$$f'(x) = \frac{(x+11)(x-1)^2}{(x+3)^3} \qquad \text{and} \qquad f''(x) = \frac{96(x-1)}{(x+3)^4}.$$

For each of the following, make sure to justify your answers.

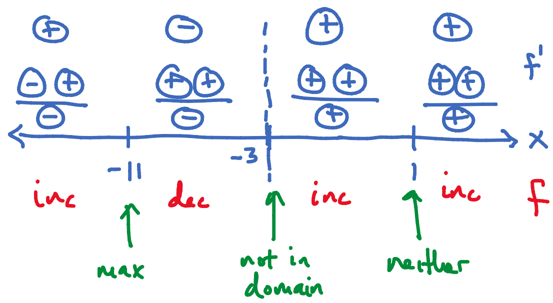
(a) Determine the interval(s) where f is increasing and decreasing.

(b) Determine the x-values(s) where any local extrema occur, or argue that there are none. Also state which type of extrema occurs.

(c) Determine the interval(s) where f is concave up and concave down.

(d) Determine the x-coordinate(s) of any inflection points, or argue that there are none.

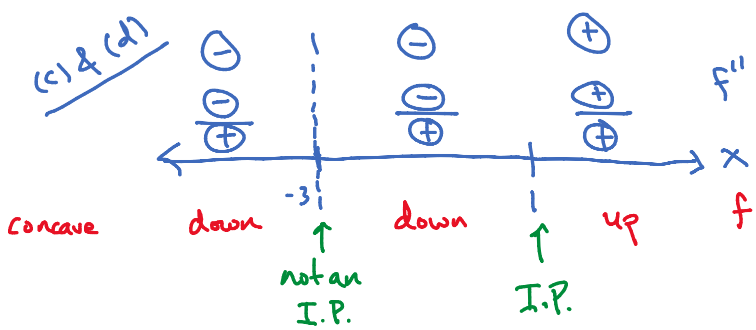
(a) & (b)



(a) increasing: $(-\infty,-11), (-3,1), (1,\infty)$
decreasing: $(-11,-3)$

(b) local max: at $x=-11$
no local mins

(c) & (d)



(c) Concave up: $(1,\infty)$
Concave down: $(-\infty,-3), (-3,1)$

(d) inflection pt at $x=1$

PART II: Multiple Choice. 4 points each

5. Find the most general antiderivative of the function $f(x) = \dfrac{8x - 2x^3\sec(x)\tan(x) + 6x^2}{x^3}$.

$= 8x^{-2} - 2\sec(x)\tan(x) + 6x^{-1}$

$\to \boxed{F(x) = -8x^{-1} - 2\sec(x) + 6\ln|x| + C}$

(a) $F(x) = -\dfrac{8}{x} - 2\sec(x) + 2x^3 + C$

(b) $F(x) = \dfrac{4x^2 - \frac{1}{2}x^4\sec(x) + 2x^3}{\frac{1}{4}x^4}$

(c) $F(x) = \dfrac{8}{x} - 2\tan(x) + 6\ln|x| + C$

(d) $F(x) = -\dfrac{8}{x} - 2\sec(x) + 6\ln|x| + C$ ← correct

(e) $F(x) = -\dfrac{8}{x} - 2\tan(x) + 2x^3 + C$

6. Use the graph of $f(x)$ below to evaluate $\int_1^8 f(x)\,dx$.

① $= 1 \times 1 = 1$

② $= 0$

③ $= 2 \times 1$

① + ② + ③ $= 1 + 0 + 2 = \boxed{3}$

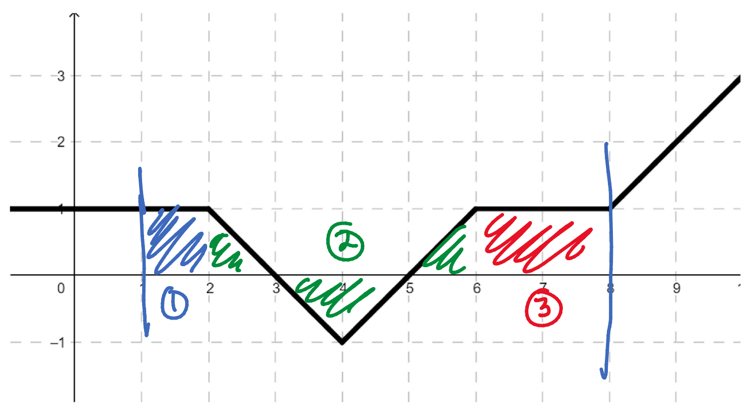
(a) 3 ← correct

(b) 7

(c) 4

(d) 5

(e) $\dfrac{9}{2}$



7. If f has domain all reals **except** $x = 3$ and $f'(x) = -\dfrac{x}{(x-3)^2}$ when $x \neq 3$, find the interval(s) where f is concave up. $\Leftrightarrow f''(x) > 0$

$f''(x) = -\left[\dfrac{(x-3)^2(1) - (x)[2(x-3)(1)]}{((x-3)^2)^2}\right] = -\left[\dfrac{(x-3)[(x-3)-2x]}{(x-3)^4}\right] = \dfrac{x+3}{(x-3)^3}$

f'': ⊕ ⊖ ⊕

x: ⊖/⊖, −3, ⊕/⊖, 3, ⊕/⊕

f: concave up, down, up

(a) $(-\infty, -3), (0, \infty)$

(b) $(-\infty, -3), (3, \infty)$ ← correct

(c) $(3, \infty)$

(d) $(0, 3)$

(e) $(-3, 3)$

8. Evaluate $\overbrace{\lim_{t\to\infty}(t \cdot g(t))}^{\infty \cdot 0 \text{ form}}$ if $g(t)$ is continuously differentiable with $\lim_{t\to\infty} g(t) = 0$ and $g'(t) = 2(1+t^2)^{-1}$.

$= \overbrace{\lim_{t\to\infty} \dfrac{g(t)}{1/t}}^{0/0 \text{ form}} = \lim_{t\to\infty} \dfrac{g'(t)}{-1/t^2} = \lim_{t\to\infty} \dfrac{2(1+t^2)^{-1}}{-1/t^2}$

$= \overbrace{\lim_{t\to\infty} \dfrac{-2t^2}{1+t^2}}^{\infty/\infty \text{ form}} = \lim_{t\to\infty} \dfrac{-4t}{2t} = \lim_{t\to\infty}(-2) = \boxed{-2}$

(a) -2 ← correct

(b) 0

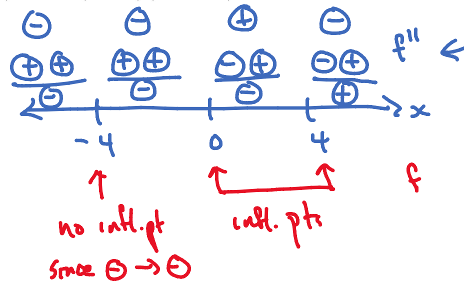
(c) ∞

(d) 2

(e) $-\infty$

$= (-x)(x+4)^2(x-4)$ ← use this for signs in f''

9. The domain of $f(x)$ is all real numbers and $f''(x) = -x(x+4)(x^2-16)$. Find the x-coordinate(s) of all inflection points for the function $f(x)$.



(a) $x = -4$ and $x = 4$

(b) $x = -4$, $x = 0$, and $x = 4$

(c) $x = -4$, $x = 0$, and $x = 16$

(d) $x = 0$ and $x = 4$ ← correct

(e) $x = -4$ and $x = 0$

10. Approximate the area under the curve $f(x) = x^3 + 30$ on the interval $[-3, 3]$ using three rectangles of equal width and left endpoints.

3 rectangles => subintervals have length $\frac{3-(-3)}{3} = 2$

=> left endpts are $-3, -1, 1$

=> approx $= 2 \cdot f(-3) + 2 \cdot f(-1) + 2 \cdot f(1)$

$= 2(3) + 2(29) + 2(31)$

$= 6 + 58 + 62 = \boxed{126}$

(a) 63

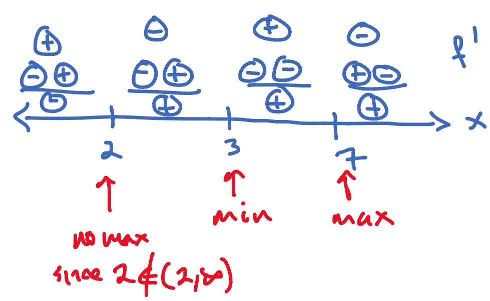
(b) 234

(c) 180

(d) 240

(e) 126 ← correct

11. The domain of $f(x)$ is $(2, \infty)$ and $f'(x) = \dfrac{(x-7)(3-x)}{x-2}$. Find the x-value(s) where the function f has as a local maximum.



(a) $x = 2$ only

(b) $x = 3$ and $x = 7$

(c) $x = 3$ only

(d) $x = 7$ only ← correct

(e) $x = 2$ and $x = 7$

12. Evaluate $\underbrace{\lim_{x \to 0} \dfrac{e^{-x} + \sin(x) - 1}{3 - 4x^2 - 3\cos(x)}}_{0/0 \text{ form}}$. $\overset{\text{L.H.}}{=} \underbrace{\lim_{x \to 0} \dfrac{-e^{-x} + \cos(x)}{-8x + 3\sin(x)}}_{0/0 \text{ form still}} \overset{\text{L.H.}}{=} \lim_{x \to 0} \dfrac{e^{x} - \sin(x)}{-8 + 3\cos(x)}$

$= \dfrac{1 - 0}{-8 + 3} = \boxed{-\dfrac{1}{5}}$

(a) $-\dfrac{1}{5}$ ← correct

(b) $-\infty$

(c) $-\dfrac{1}{3}$

(d) $-\dfrac{1}{4}$

(e) 0

13. Find the absolute minimum and maximum values of the function $f(x) = 2x^3 + 6x^2 - 18x$ on the interval $[-1, 2]$.

(a) 0, 22
(b) -10, 0
(c) -10, 54
(d) -10, 22 ← correct
(e) 4, 22

$f'(x) = 6x^2 + 12x - 18 = 6(x^2+2x-3) = 6(x+3)(x-1) = 0$

$\Rightarrow x = \cancel{-3}, 1$

not in $[-1,2]$

largest

$f(-1) = -2+6+18 = \boxed{22}$ $\quad f(1) = 2+6-18 = \boxed{-10}$ smallest

$f(2) = 16+24-36 = 4$

14. The acceleration of a particle is given by $a(t) = 12t^2 - 7\sin(t)$ with $v(0) = -3$ and $s(0) = 5$. Find the position function for the particle.

(a) $s(t) = t^4 + 7\sin(t) - 3t + 5$
(b) $s(t) = t^4 - 7\sin(t) - 10t + 5$
(c) $s(t) = t^4 + 7\sin(t) - 17t + 5$
(d) $s(t) = t^4 - 7\sin(t) - 3t + 5$
(e) $s(t) = t^4 + 7\sin(t) - 10t + 5$ ← correct

$v(t) = 4t^3 + 7\cos(t) + C$

$-3 = v(0) = 0 + 7 + C \Rightarrow C = -10$

$\Rightarrow s(t) = t^4 + \sin(t) - 10t + D$

$5 = s(0) = D \Rightarrow D = 5$

$\Rightarrow \boxed{s(t) = t^4 + \sin(t) - 10t + 5}$

15. Find the interval(s) where the function $f(x) = e^x(x^2 - 7x + 8)$ is concave downwards.

(a) $(0, 4)$
(b) $(-1, 4)$ ← correct
(c) $(-\infty, -1), (0, \infty)$
(d) $(-\infty, -1)$
(e) $(-\infty, -1), (4, \infty)$

$\Rightarrow f'(x) = e^x(x^2-7x+8) + e^x(2x-7) = e^x(x^2-5x+1)$

$\Rightarrow f''(x) = e^x(x^2-5x+1) + e^x(2x-5) = e^x(x^2-3x-4) = e^x(x-4)(x+1)$

⊕ ⊖ ⊕ f''

⊖⊖ ⊖⊕ ⊕⊕

x: -1, 4

note $e^x > 0$ for all x

so only look at the factors

conc. up down up f

16. Which of the following gives the exact net area under the curve $f(x) = \cos(x)$ on the interval $[1, 6]$?

(a) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \cos\left(1 + \frac{6i}{n}\right)$

(b) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n} \cos\left(1 + \frac{5i}{n}\right)$ ← correct

(c) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \cos\left(\frac{6i}{n}\right)$

(d) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n} \cos\left(\frac{5i}{n}\right)$

(e) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \cos\left(1 + \frac{5i}{n}\right)$

$\Delta x = \frac{b-a}{n} = \frac{6-1}{n} = \frac{5}{n}$

right endpts $\Rightarrow x_i^* = a + i\Delta x = 1 + i\cdot\frac{5}{n} = 1 + \frac{5i}{n}$

17. Find a number c that satisfies the conclusion of the Mean Value Theorem for the function $f(x) = \ln(x) - x$ on the interval $[1, e]$.

(a) $c = 0$

(b) $c = e$

(c) $c = e - 1$ ← correct

(d) $c = \dfrac{1}{e-1}$

(e) $c = 1$

MVT: want c s.t. $f'(c) = \dfrac{f(e)-f(1)}{e-1}$

$$\Leftrightarrow \frac{1}{c} - 1 = \frac{(1-e)-(0-1)}{(e-1)} = -1 + \frac{1}{e-1}$$

$$\Rightarrow \boxed{c = e-1}$$

18. Given that $\int_5^1 f(x)\,dx = -2$, $\int_3^5 g(x)\,dx = 9$, and $\int_3^1 g(x)\,dx = 7$, determine the value of $\int_1^5 (4f(x) - g(x))$.

(a) -24

(b) 0

(c) 14

(d) -8

(e) 6 ← correct

$$\int_1^5 (4f(x)-g(x))dx = 4\underbrace{\int_1^5 f(x)\,dx} - \underbrace{\int_1^5 g(x)\,dx} = 4(2) - 2 = \boxed{6}$$

$$= -\int_5^1 f(x)\,dx = -(-2) = 2$$

$$= \int_1^3 g(x)\,dx + \int_3^5 g(x)\,dx = -\int_3^1 g(x)\,dx + \int_3^5 g(x)\,dx$$

$$= -7 + 9 = 2$$

The graph below is the **derivative**, $f'(x)$, of a continuous function f whose domain is all real numbers. Use this graph to answer Questions 19 and 20.

19. On what interval(s) is $f(x)$ decreasing?

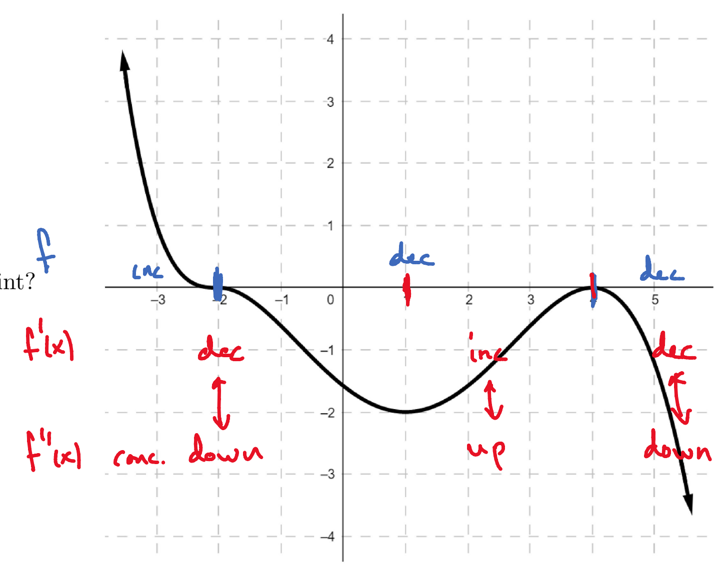
(a) $(-\infty, -2), (-2, 4), (4, \infty)$

(b) $(-\infty, -2)$

(c) $(-2, 4), (4, \infty)$ ← correct

(d) $(-\infty, 1), (4, \infty)$

(e) $(1, 4)$



20. For what x-value(s) does f have an inflection point?

(a) $x = -2$ only

(b) $x = -2$ and $x = 4$

(c) $x = 1$ and $x = 4$ ← correct

(d) $x = 1$ only

(e) $x = -2$, $x = 1$, and $x = 4$