MATH151, Fall 2022
Common Exam I - Version A Solutions

LAST NAME (print): _____ FIRST NAME (print): _____

INSTRUCTOR: _____

SECTION NUMBER: _____

DIRECTIONS:

- No calculators, cell phones, smart watches, headphones, or other electronic devices may be used, and must be put away.

- TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.

- In Part I, mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*

- In Part II, present your solutions in the space provided. Show all your work neatly and concisely, and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

- **Be sure to fill in your name, UIN, section number, and version letter of the exam on the ScanTron form.**

THE AGGIE HONOR CODE

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: _____

Part I: Multiple Choice. 3 points each

1. Find a vector of length 3 in the same direction as the vector from the point $(-2, 3)$ to $(1, -1)$.

   (a) $\left\langle \dfrac{9}{5}, -\dfrac{12}{5} \right\rangle$ ← correct

   (b) $\left\langle -\dfrac{9}{5}, \dfrac{12}{5} \right\rangle$

   (c) $\left\langle \dfrac{9}{\sqrt{5}}, -\dfrac{12}{\sqrt{5}} \right\rangle$

   (d) $\left\langle -\dfrac{3}{\sqrt{5}}, \dfrac{6}{\sqrt{5}} \right\rangle$

   (e) $\left\langle -\dfrac{3}{\sqrt{17}}, -\dfrac{12}{\sqrt{17}} \right\rangle$

   Solution. Consider $A(-2, 3)$ and $B(1, -1)$:

   $$\overrightarrow{AB} = \langle 1 - (-2),\ -1 - 3 \rangle = \langle 3, -4 \rangle \implies \mathbf{u} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{\langle 3, -4 \rangle}{\sqrt{9 + 16}} = \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle \implies 3\mathbf{u} = \left\langle \frac{9}{5}, \frac{-12}{5} \right\rangle$$

2. Find a vector equation of the line that passes through $(-6, 4)$ and is perpendicular to the line with parametric equations $x = 7 + 2t$, $y = 1 - 3t$.

   (a) $\mathbf{r}(t) = \langle 4 + 3t,\ -6 + 2t \rangle$
   (b) $\mathbf{r}(t) = \langle -6 - t,\ 4 + 7t \rangle$
   (c) $\mathbf{r}(t) = \langle -6 + 2t,\ 4 - 3t \rangle$
   (d) $\mathbf{r}(t) = \langle 4 + 2t,\ -6 - 3t \rangle$
   (e) $\mathbf{r}(t) = \langle -6 + 3t,\ 4 + 2t \rangle$ ← correct

   Solution. We have $\mathbf{r}_0 = \langle -6, 4 \rangle$ and $\mathbf{v} = \langle 2, -3 \rangle^{\perp} = \langle 3, 2 \rangle$, so $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \langle -6 + 3t,\ 4 + 2t \rangle$.

3. Find $\displaystyle\lim_{x \to \infty} \left[\ln(5x^2 + 1) - \ln(x^3 + x - 4) \right]$.

   (a) ∞
   (b) $-\infty$ ← correct
   (c) 0
   (d) $\ln(5)$
   (e) $\dfrac{2}{3}$

   Solution. $\displaystyle\lim_{x \to \infty} \left[\ln(5x^2 + 1) - \ln(x^3 + x - 4) \right] = \lim_{x \to \infty} \ln\left(\frac{5x^2 + 1}{x^3 + x - 4} \right)$, where

   $$\lim_{x \to \infty} \frac{5x^2 + 1}{x^3 + x - 4} = \lim_{x \to \infty} \frac{5x^2}{x^3} = \lim_{x \to \infty} \frac{5}{x} = 0^+$$

   so $\displaystyle\lim_{x \to \infty} \ln\left(\frac{5x^2 + 1}{x^3 + x - 4} \right)^{\nearrow 0^+} = -\infty.$

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4. Find the cosine of the angle between the vectors $\langle -3, 5 \rangle$ and $\langle -2, 1 \rangle$.

   (a) $\dfrac{11}{\sqrt{34}\sqrt{5}}$ ← correct

   (b) $\dfrac{11}{\sqrt{2}}$

   (c) $\dfrac{11}{4\sqrt{5}}$

   (d) $-\dfrac{1}{\sqrt{34}\sqrt{5}}$

   (e) $\dfrac{30}{\sqrt{34}\sqrt{5}}$

   Solution. $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{6+5}{\sqrt{34}\sqrt{5}} = \dfrac{11}{\sqrt{34}\sqrt{5}}$

5. Find the scalar projection of $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{a} = 2\mathbf{i} - 5\mathbf{j}$.

   (a) $-\dfrac{4}{\sqrt{13}}$

   (b) $\dfrac{2}{\sqrt{34}}$

   (c) $-\dfrac{4}{\sqrt{29}}$ ← correct

   (d) $\dfrac{4}{\sqrt{29}}$

   (e) $\dfrac{4}{\sqrt{13}}$

   Solution. $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{6-10}{\sqrt{4+25}} = \dfrac{-4}{\sqrt{29}}$

6. Find $\displaystyle\lim_{x\to-\infty} \dfrac{2x+3}{\sqrt{9x^2 - 2x + 5}}$.

   (a) $\dfrac{2}{3}$

   (b) $-\dfrac{2}{9}$

   (c) $\dfrac{2}{9}$

   (d) $-\dfrac{2}{3}$ ← correct

   (e) 0

   Solution.
   $$\lim_{x\to-\infty} \frac{2x+3}{\sqrt{9x^2 - 2x + 5}} = \lim_{x\to-\infty} \frac{2x}{\sqrt{9x^2}} = \lim_{x\to-\infty} \frac{2x}{3|x|} = \lim_{x\to-\infty} \frac{2x}{3(-x)} = -\frac{2}{3}$$

7. Which of the following is equal to $\tan\left(\arccos\left(\frac{x}{2}\right)\right)$?

(a) $\dfrac{\sqrt{4+x^2}}{x}$

(b) $\dfrac{\sqrt{4-x^2}}{x}$ $\leftarrow$ correct

(c) $\dfrac{\sqrt{4-x^2}}{2}$

(d) $\dfrac{x}{\sqrt{4-x^2}}$

(e) $\dfrac{2}{\sqrt{4-x^2}}$

Solution. Let $\theta = \arccos\left(\frac{x}{2}\right) \implies \cos\theta = \dfrac{x}{2} \ \left[\dfrac{Adj}{Hyp}\right]$. This gives a reference triangle with $Opp = \sqrt{4-x^2}$, so $\tan\left(\arccos\left(\frac{x}{2}\right)\right) = \tan\theta = \dfrac{\sqrt{4-x^2}}{x} \ \left[\dfrac{Opp}{Adj}\right]$.

8. Find $\displaystyle\lim_{x\to2^-} \dfrac{3x+6}{x-2}$.

(a) ∞

(b) 0

(c) 3

(d) 12

(e) $-\infty$ $\leftarrow$ correct

Solution. Note that $x < 2$, so $x - 2 < 0$: $\displaystyle\lim_{x\to2^-} \dfrac{3x+6}{x-2} = \text{``}\dfrac{12}{0^-}\text{''} = -\infty$

9. The position (in meters) of a particle after t seconds is given by $f(t) = (t+1)^2 - 1$. Find the average velocity of the particle from $t = 1$ to $t = 3$.
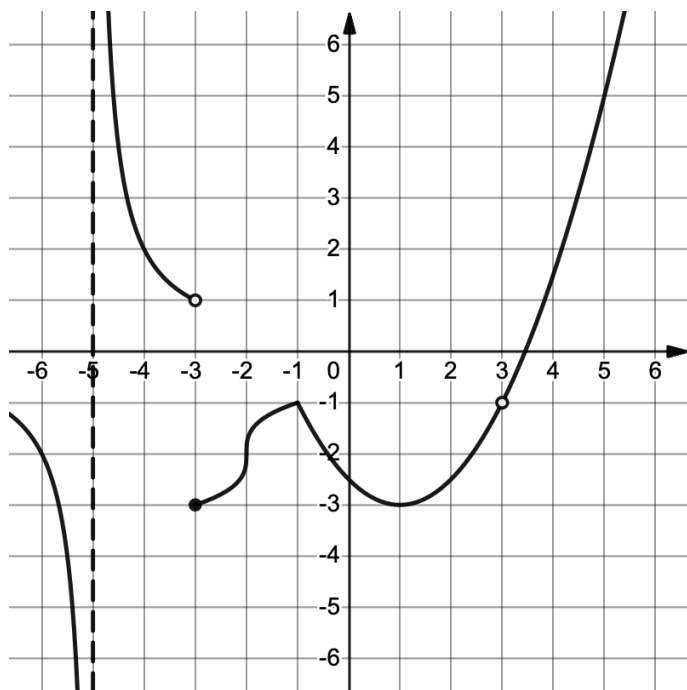
(a) 8 m/s

(b) 4 m/s

(c) 6 m/s $\leftarrow$ correct

(d) $\dfrac{26}{3}$ m/s

(e) 9 m/s

Solution. $\dfrac{f(3) - f(1)}{3 - 1} = \dfrac{15 - 3}{2} = 6$

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Use the following graph of f to answer questions 10 and 11.

10. Given the graph of f above, which of the following is **false**?

    (a) $\lim\limits_{x \to 5} f(x) = 5$.

    (b) $\lim\limits_{x \to 3} f(x) = -1$.

    (c) f is continuous from the left at $x = -3$. ← correct

    (d) $\lim\limits_{x \to -5^+} f(x) = \infty$.

    (e) f is continuous at $x = -1$.

    Solution. $\lim\limits_{x \to -3^-} f(x) = 1$, but $f(-3) = -3$

11. Given the graph of f above, for which value of x is $f(x)$ differentiable?

    (a) $x = -3$

    (b) $x = 3$

    (c) $x = -1$

    (d) $x = 1$ ← correct

    (e) $x = -2$

    Solution. f has a horizontal tangent line at $x = 1$, so $f'(1) = 0$. (discontinuities at $x = -3, 3$; corner at $x = -1$; vertical tangent at $x = -2$)

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12. A crate is hauled 10 meters up a ramp under a constant force of 7 Newtons applied at an angle of $30°$ to the ramp. Find the work done.

   (a) $35\sqrt{2}$ J

   (b) 35 J

   (c) 70 J

   (d) $35\sqrt{3}$ J ← correct

   (e) None of these.

   Solution. $W = \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos\theta = (7)(10)\cos(30°) = 70(\sqrt{3}/2) = 35\sqrt{3}$

13. Find the value of a that makes $f(x) = \begin{cases} ax^2 - 2x & \text{if } x \leq 2 \\ 3x + a & \text{if } x > 2 \end{cases}$ continuous everywhere.

   (a) 2

   (b) $\dfrac{10}{3}$ ← correct

   (c) $\dfrac{2}{3}$

   (d) 4

   (e) There is no such a value.

   Solution. We need $\lim\limits_{x \to 2} f(x) = f(2)$:

   $$f(2) = 4a - 4$$
   $$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (ax^2 - 2x) = 4a - 4$$
   $$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x + a) = 6 + a$$
   $$\implies 4a - 4 = 6 + a$$
   $$\implies a = 10/3$$

14. According to the Intermediate Value Theorem, which of the following intervals contains a solution to the equation $x^3 - 4x - 5 = 0$?

   (a) $(-2, -1)$

   (b) $(0, 1)$

   (c) $(-1, 0)$

   (d) $(2, 3)$ ← correct

   (e) $(1, 2)$

   Solution. We have $f(2) < 0 < f(3)$. (Note that $f(-2) = -5$, $f(-1) = -2$, $f(0) = -5$, $f(1) = -8$, $f(2) = -5$, $f(3) = 10$)

15. Find $\displaystyle\lim_{x\to\infty} \frac{4x - \sqrt{x^2 + 5x}}{3x}$.

   (a) $\dfrac{4}{3}$

   (b) $\dfrac{5}{3}$

   (c) 1 ← correct

   (d) ∞

   (e) $-\infty$

   Solution. $\displaystyle\lim_{x\to\infty} \frac{4x - \sqrt{x^2 + 5x}}{3x} = \lim_{x\to\infty} \frac{4x - |x|}{3x} = \lim_{x\to\infty} \frac{4x - x}{3x} = \lim_{x\to\infty} \frac{3x}{3x} = 1$

16. Find $\displaystyle\lim_{x\to-\infty} \frac{5e^{-2x} + 3e^{4x}}{2e^{-2x} - e^{4x}}$.

   (a) -3

   (b) 0

   (c) ∞

   (d) $\dfrac{5}{2}$ ← correct

   (e) $-\infty$

   Solution. $\displaystyle\lim_{x\to-\infty} \frac{5e^{-2x} + 3e^{4x}}{2e^{-2x} - e^{4x}} = \lim_{x\to-\infty} \frac{5e^{-2x}}{2e^{-2x}} = \frac{5}{2}$ (Note that $\displaystyle\lim_{x\to-\infty} e^{-2x} = \infty$ and $\displaystyle\lim_{x\to-\infty} e^{4x} = 0$)

17. Which of the following correctly describes the curve given by $\mathbf{r}(t) = \langle 1 + \sin t, \ -2 + \cos t\rangle$ as t increases?

   (a) Clockwise around a circle with center $(1, -2)$. ← correct
   (b) Counterclockwise around a circle with center $(1, -2)$.
   (c) Clockwise around a circle with center $(-1, 2)$.
   (d) Counterclockwise around a circle with center $(-1, 2)$.
   (e) None of these.

   Solution. $x = 1 + \sin t \implies \sin t = x - 1$ and $y = -2 + \cos t \implies \cos t = y + 2$.

$$\sin^2 t + \cos^2 t = 1 \implies (x-1)^2 + (y+2)^2 = 1$$

   This is a circle of radius 1 with center $(1, -2)$. We can plot some points to see clockwise behavior:

t	x	y
0	1	-1
$\pi/2$	2	-2
π	1	-3

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18. Find the horizontal and vertical asymptotes of $f(x) = \dfrac{x^2 - 4}{(2-x)(x+3)}$.

    (a) $y = -1$, $x = 2$, $x = -3$

    (b) $y = 1$, $x = -3$

    (c) $y = -1$, $x = -3$ ← correct

    (d) $y = 1$, $x = 2$

    (e) $y = -1$, $y = 1$, $x = -3$

Solution. For horizontal asymptotes, we want x-values that give infinite limits (nonzero/0):

$$\frac{x^2 - 4}{(2-x)(x+3)} = \frac{(x+2)(x-2)}{-(x-2)(x+3)} \to \frac{(x+2)}{-(x+3)}$$

Thus $x = -3$ is a horizontal asymptote. For vertical asymptotes, we need y-values corresponding to limits at infinity:

$$\frac{x^2 - 4}{(2-x)(x+3)} = \frac{x^2 - 4}{-x^2 - x + 6} \xrightarrow{x \to \pm\infty} = \frac{x^2}{-x^2} = -1$$

Thus $y = -1$ is a vertical asymptote.

19. Find $\displaystyle\lim_{x \to \infty} \arctan\left(\dfrac{x - x^3}{x + 7}\right)$.

    (a) $\dfrac{\pi}{2}$

    (b) $-\dfrac{\pi}{4}$

    (c) $\dfrac{\pi}{4}$

    (d) $-\infty$

    (e) $-\dfrac{\pi}{2}$ ← correct

Solution. $\displaystyle\lim_{x\to\infty} \frac{x - x^3}{x+7} = \lim_{x\to\infty} \frac{-x^3}{x} = \lim_{x\to\infty} -x^2 = -\infty \implies \lim_{x\to\infty} \arctan\left(\frac{x-x^3}{x+7}\right) = -\frac{\pi}{2}$

20. Given $f(x) = x^3 - 4x + 1$ and $f'(x) = 3x^2 - 4$, find the equation of the tangent line to $f(x)$ at $x = -2$.

    (a) $y = 8x - 10$

    (b) $y = 8x + 1$

    (c) $y = x + 10$

    (d) $y = 8x + 17$ ← correct

    (e) $y = x + 3$

Solution. $f(-2) = -8 + 8 + 1 = 1$, $f'(-2) = 12 - 4 = 8 \implies y - 1 = 8(x+2) \implies y = 8x + 17$

<u>**Directions**</u>: Present your solutions in the space provided. *Show all your work* neatly and concisely and *box your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

21. (10 points) Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on an object. The force $\mathbf{F}_1$ has a magnitude of 26 lbs and a direction of $45°$ counterclockwise from the positive x-axis, and $\mathbf{F}_2$ has a magnitude of 8 lbs and a direction of $150°$ counterclockwise from the positive x-axis.

   (a) Find the resultant force $\mathbf{F}$.

   Solution.

   $$\mathbf{F}_1 = |\mathbf{F}_1|\langle \cos(45°), \sin(45°)\rangle = 26 \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle = \left\langle 13\sqrt{2},\ 13\sqrt{2} \right\rangle$$

   $$\mathbf{F}_2 = |\mathbf{F}_2|\langle \cos(150°), \sin(150°)\rangle \qquad \left(= |\mathbf{F}_2|\langle -\cos(30°), \sin(30°)\rangle \right)$$

   $$= 8 \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

   $$= \left\langle -4\sqrt{3},\ 4 \right\rangle$$

   $$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \boxed{\left\langle 13\sqrt{2} - 4\sqrt{3},\ 13\sqrt{2} + 4 \right\rangle}$$

   (b) Find the resultant angle θ as measured counterclockwise from the positive x-axis. Leave your answer in terms of an inverse trigonometric expression.

   Solution.

   $$\tan\theta = \frac{13\sqrt{2} + 4}{13\sqrt{2} - 4\sqrt{3}} \implies \theta = \boxed{\tan^{-1}\left(\frac{13\sqrt{2} + 4}{13\sqrt{2} - 4\sqrt{3}} \right)}$$

22. (15 points) Find the following limits. *Do not use L'Hospital's Rule.*

(a) $\lim\limits_{x \to 6} \dfrac{(x-2)^2 - 16}{x^2 - 6x}$

Solution.

$$\lim_{x \to 6} \frac{(x-2)^2 - 16}{x^2 - 6x} = \lim_{x \to 6} \frac{x^2 - 4x + 4 - 16}{x(x-6)} = \lim_{x \to 6} \frac{x^2 - 4x - 12}{x(x-6)} = \lim_{x \to 6} \frac{(x-6)(x+2)}{x(x-6)}$$

$$= \lim_{x \to 6} \frac{x+2}{x}$$

$$= \frac{8}{6} = \boxed{\frac{4}{3}}$$

(b) $\lim\limits_{x \to 2} \dfrac{\sqrt{5x+6} - 4}{x - 2}$

Solution.

$$\lim_{x \to 2} \frac{\sqrt{5x+6} - 4}{x - 2} = \lim_{x \to 2} \frac{\sqrt{5x+6} - 4}{x - 2} \cdot \frac{\sqrt{5x+6} + 4}{\sqrt{5x+6} + 4}$$

$$= \lim_{x \to 2} \frac{(5x+6) - 16}{(x-2)(\sqrt{5x+6} + 4)}$$

$$= \lim_{x \to 2} \frac{5x - 10}{(x-2)(\sqrt{5x+6} + 4)}$$

$$= \lim_{x \to 2} \frac{5(x-2)}{(x-2)(\sqrt{5x+6} + 4)}$$

$$= \lim_{x \to 2} \frac{5}{\sqrt{5x+6} + 4}$$

$$= \boxed{\frac{5}{8}}$$

(c) $\lim\limits_{x \to -1^-} \dfrac{|x+1|}{x^2 + 3x + 2}$

Solution. Since $x < -1$, $|x+1| = -(x+1)$:

$$\lim_{x \to -1^-} \frac{|x+1|}{x^2 + 3x + 2} = \lim_{x \to -1^-} \frac{-(x+1)}{(x+1)(x+2)} = \lim_{x \to -1^-} \frac{-1}{x+2} = \frac{-1}{-1+2} = \boxed{-1}$$

23. (6 points) Consider the function $f(x) = \begin{cases} 2e^{x+3} & \text{if } x \leq -3 \\ 3x - 1 & \text{if } -3 < x < 1 \\ \sqrt{x^2 + 3} & \text{if } x > 1 \end{cases}$

(a) Find $\lim\limits_{x \to -3} f(x)$, or explain why it does not exist.

Solution.

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} 2e^{x+3} = 2e^0 = 2$$

and

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (3x - 1) = -9 - 1 = -10$$

These are not equal, so $\lim\limits_{x \to -3} f(x)$ does not exist (DNE).

(b) Find $\lim\limits_{x \to 1^+} f(x)$, or explain why it does not exist.

Solution.
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x^2 + 3} = \sqrt{1 + 3} = \boxed{2}$$

24. (9 points) Use the **definition** of the derivative to find $f'(x)$ for $f(x) = \dfrac{4}{x - 2}$. *No points will be given for any shortcut formulas used!*

Solution.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{4}{(x+h) - 2} - \dfrac{4}{x - 2}}{h} \cdot \frac{(x+h-2)(x-2)}{(x+h-2)(x-2)}$$

$$= \lim_{h \to 0} \frac{4(x - 2) - 4(x + h - 2)}{h(x + h - 2)(x - 2)}$$

$$= \lim_{h \to 0} \frac{4x - 8 - 4x - 4h + 8}{h(x + h - 2)(x - 2)}$$

$$= \lim_{h \to 0} \frac{-4h}{h(x + h - 2)(x - 2)}$$

$$= \lim_{h \to 0} \frac{-4}{(x + h - 2)(x - 2)}$$

$$= \boxed{\frac{-4}{(x - 2)^2}}$$

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Do not write in this table.

Question	Points Awarded	Points
1-20		60
21		10
22		15
23		6
24		9
Total		100