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Fall 2021

MATH 152, FALL 2021
COMMON EXAM II - VERSION A KEY

LAST NAME(print): _____ FIRST NAME(print): _____

INSTRUCTOR: _____

SECTION NUMBER: _____

DIRECTIONS:

- The use of a calculator, laptop or computer is prohibited.
- TURN OFF all phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
- In Part 1 (Problems 1-15), mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore for your own records, also record your choices on your exam!
- In Part 2 (Problems 16-19), present your solutions in the space provided. Show all your work neatly and concisely and clearly indicate your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
- Be sure to write your name, section number and version letter of the exam on the ScanTron form.
- Again. The use of a calculator, laptop or computer is prohibited.

THE AGGIE HONOR CODE
"An Aggie does not lie, cheat, or steal, or tolerate those who do."
Signature: _____

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Question	Points Awarded	Points
1-15	ScanTron	60
16		12
17		12
18		8
19		8
TOTAL		100

Part 1: Multiple Choice (4 points each)

1. Which statement is true about the integral $\int_{-2}^1 \frac{2}{(x-3)^2} dx$?
- (a) Diverges **key**
 (b) Converges to $\frac{2}{3}$
 (c) Converges to $\frac{1}{3}$
 (d) Converges to $-\frac{2}{3}$
 (e) Converges to $-\frac{1}{3}$
- $$= \frac{2}{x-3} \Big|_{-2}^1 = \frac{2}{1-3} - \frac{2}{-2-3} = -1 - (-\frac{2}{5}) = -\frac{3}{5}$$

2. Which sequence is both bounded and increasing?
- (a) $a_n = 1 - \frac{2}{n}$ **key**
 (b) $a_n = \ln n$: increasing but **Not bounded**
 (c) $a_n = \sin(2n\pi)$: bounded but **Not monotonic**
 (d) $a_n = e^{-n}$: decreasing
 (e) None of these

3. The integral $\int_0^{\infty} e^{-2x} dx$
- (a) diverges
 (b) converges to 0
 (c) converges to $\frac{1}{2}$ **key**
 (d) converges to $\frac{1}{2}$ **key**
 (e) converges to 2
- $$= \frac{1}{-2} e^{-2x} \Big|_0^{\infty} = \frac{1}{-2} (0 - 1) = \frac{1}{2}$$

4. Which of the following integrals is equivalent to $\int \sqrt{4x^2 - 9} dx$?
- (a) $\int \sec \theta \tan^2 \theta d\theta$
 (b) $\int \tan \theta d\theta$
 (c) $\int \sec \theta \tan^2 \theta d\theta$ **key**
 (d) $\int \sec^2 \theta \tan \theta d\theta$
 (e) $\int \sec^2 \theta \tan \theta d\theta$
- Let $x = \frac{3}{2} \sec \theta$
 $dx = \frac{3}{2} \sec \theta \tan \theta d\theta$
- Then $\int \sqrt{4x^2 - 9} dx = \int \sqrt{9 \sec^2 \theta - 9} \cdot \frac{3}{2} \sec \theta \tan \theta d\theta$
 $= \int 3 \tan \theta \cdot \frac{3}{2} \sec \theta \tan \theta d\theta = \frac{9}{2} \int \sec \theta \tan^2 \theta d\theta$

5. After an appropriate substitution, the integral $\int \sqrt{9 - 4x^2} dx$ is equivalent to which of the following?
- (a) $\int \sec \theta \tan^2 \theta d\theta$
 (b) $\int \cos \theta d\theta$
 (c) $\int \sec^2 \theta d\theta$
 (d) $\int \cos^2 \theta d\theta$ **key**
 (e) $\int \tan \theta d\theta$
- Let $x = \frac{3}{2} \sin \theta$
 $dx = \frac{3}{2} \cos \theta d\theta$
- Then $\int \sqrt{9 - 4x^2} dx = \int \sqrt{9 - 9 \sin^2 \theta} \cdot \frac{3}{2} \cos \theta d\theta$
 $= \int 3 \cos \theta \cdot \frac{3}{2} \cos \theta d\theta = \frac{9}{2} \int \cos^2 \theta d\theta$

6. Determine whether the following series converges or diverges. If it converges, find the sum.
- (a) Diverges
 (b) Converges to 0
 (c) Converges to 1
 (d) Converges to $\frac{1}{2}$
 (e) Converges to 3 **key**
- $$\sum_{n=1}^{\infty} \left(\frac{3}{n+1} \right)$$
- $$= \lim_{n \rightarrow \infty} \frac{3}{n+1} = \frac{3}{\infty} = 0$$

7. Evaluate $\int_0^1 \frac{4x^2 + 5}{2x + 1} dx$
- (a) $2 \ln 3$
 (b) $3 \ln 3$ **key**
 (c) $\ln 3$
 (d) $6 \ln 3$
 (e) None of these
- long division: $\frac{2x+1}{4x^2+5} = \frac{2x-1}{4x^2+5} + \frac{2}{2x+1}$
- $$= \int_0^1 \left(2x - 1 + \frac{6}{2x+1} \right) dx = \left[x^2 - x + 3 \ln|2x+1| \right]_0^1 = (1 - 1 + 3 \ln 3) - (0 - 0 + 3 \ln 1) = 3 \ln 3$$

8. Write out the form of the partial fraction decomposition of the function
- $f(x) = \frac{x^3 - 2x^2 - 5x + 4}{(x+2)^2(x^2 - 4x + 7)}$
- (a) $\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1} + \frac{D}{x+1} + \frac{Ez+F}{x^2+5z+7}$
 (b) $\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cz+D}{x^2-4z+7} + \frac{Ez+F}{x^2+5z+7}$
 (c) $\frac{A}{(x+2)^2} + \frac{B}{x-1} + \frac{C}{x+1} + \frac{Dz+E}{x^2+5z+7}$
 (d) $\frac{A}{(x+2)^2} + \frac{B}{x^2-4} + \frac{Cz+D}{x^2+5z+7}$
 (e) $\frac{Ax+B}{(x+2)^2} + \frac{Cx+D}{x^2-4} + \frac{Ez+F}{x^2+5z+7}$

9. The recursive sequence given below is bounded and increasing. Determine whether the sequence converges or diverges. If it converges, find the limit of the sequence.
- $a_1 = 4, a_{n+1} = 8 - \frac{15}{a_n}$
- (a) The sequence diverges.
 (b) 3
 (c) 4
 (d) 5 **key**
 (e) 8
- Since the sequence is bounded and monotonic, it converges by the Monotone Sequence Theorem.
 And let $\lim_{n \rightarrow \infty} a_n = L$, then $\lim_{n \rightarrow \infty} a_{n+1} = L$ too.
 $\Rightarrow L = 8 - \frac{15}{L} \Rightarrow L^2 = 8L - 15$
 $\Rightarrow L^2 - 8L + 15 = 0$
 $\Rightarrow (L-3)(L-5) = 0$
 $\therefore L = 3$ or 5
 Since $a_1 = 4 > 3$, $L = 5$ is the limit.

10. Which statement is true about the integral $\int_1^{\infty} \frac{3 \sin^2 x}{x^2} dx$?
- (a) The integral converges by comparison to $\int_1^{\infty} \frac{1}{x^2} dx$
 (b) The integral diverges by comparison to $\int_1^{\infty} \frac{1}{x^2} dx$
 (c) The integral converges by comparison to $\int_1^{\infty} \frac{3}{x^2} dx$ **key**
 (d) The integral diverges by comparison to $\int_1^{\infty} \frac{1}{x} dx$
 (e) None of these
- Since $0 \leq \sin^2 x \leq 1$,
 $\frac{3 \sin^2 x}{x^2} \leq \frac{3}{x^2}$
 And since $\int_1^{\infty} \frac{3}{x^2} dx$ converges,
 $\int_1^{\infty} \frac{3 \sin^2 x}{x^2} dx$ converges by the Comparison Theorem.

11. Consider the series $\sum_{n=1}^{\infty} a_n$ whose n-th partial sum is given by $s_n = \frac{2}{3 - e^{-2n}}$. What is $\sum_{n=1}^{\infty} a_n$?
- (a) 0
 (b) $\frac{1}{2}$
 (c) 1
 (d) 2 **key**
 (e) 3
- $$s_n = \frac{2}{3 - e^{-2n}} \Rightarrow a_n = s_n - s_{n-1} = \frac{2}{3 - e^{-2n}} - \frac{2}{3 - e^{-2(n-1)}}$$
- $$= \frac{2}{3 - e^{-2n}} - \frac{2}{3 - e^{-2n+2}} = \frac{2}{3 - e^{-2n}} - \frac{2e^2}{3 - e^{-2n}}$$

12. Which of the following integrals is equivalent to $\int \frac{1}{(x^2 - 4x + 5)^{3/2}} dx$?
- (a) $\int \frac{1}{9} \cos \theta d\theta$
 (b) $\int \cos^2 \theta d\theta$
 (c) $\int \frac{1}{27} \cos^3 \theta d\theta$
 (d) $\int \sec \theta d\theta$
 (e) $\int \cos \theta d\theta$ **key**
- Let $x = \tan \theta + 2 \Rightarrow dx = \sec^2 \theta d\theta$
- $$= \int \frac{1}{(\tan^2 \theta + 1)^{3/2}} \cdot \sec^2 \theta d\theta = \int \frac{1}{(\sec^2 \theta)^{3/2}} \cdot \sec^2 \theta d\theta = \int \frac{1}{\sec \theta} d\theta = \int \cos \theta d\theta$$

13. Use the Remainder Estimate for the Integral Test to determine the minimum number of terms needed to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{2n^2}$ to within $\frac{1}{80}$.
- (a) 4 terms
 (b) 5 terms
 (c) 6 terms
 (d) 7 terms **key**
 (e) 8 terms
- $R_n \leq \int_n^{\infty} \frac{1}{2x^2} dx = \frac{1}{2n} < \frac{1}{80}$
- $\Rightarrow \frac{1}{2n} < \frac{1}{80} \Rightarrow n > 40$
 \therefore the smallest n is 7.

14. The integral $\int_0^1 \ln x dx$
- (a) diverges
 (b) converges to 0
 (c) converges to e
 (d) converges to 1
 (e) converges to -1 **key**
- By parts, $u = \ln x, dv = dx$
 $du = \frac{1}{x} dx, v = x$
- $$= \frac{1}{e} \left[x \ln x - \int \frac{1}{x} dx \right]_0^1 = \frac{1}{e} \left[(1 \cdot 0 - 1) - \lim_{x \rightarrow 0^+} (x \ln x - 1) \right]$$
- $$= \frac{1}{e} \left[-1 - \lim_{x \rightarrow 0^+} (x \ln x - 1) \right] = \frac{1}{e} \left[-1 - (-1) \right] = 0$$

15. Which of the following sequences converges?
- (i) $a_n = \cos \left(\frac{1}{n} \right)$ (ii) $a_n = \frac{(-1)^n 3n}{n+1}$ (iii) $a_n = \ln(n^2 + 1) - \ln n$
- (a) Only (i) converges **key**
 (b) Only (ii) converges
 (c) Only (i) and (iii) converge
 (d) Only (i) and (ii) converge
 (e) All three sequences diverge
- (i) $\lim_{n \rightarrow \infty} \cos \left(\frac{1}{n} \right) = \cos(0) = 1$: converges
 (ii) $\lim_{n \rightarrow \infty} \frac{(-1)^n 3n}{n+1} = \lim_{n \rightarrow \infty} (-1)^n \frac{3n}{n+1} = 3$: oscillation : diverges
 (iii) $\lim_{n \rightarrow \infty} \ln(n^2 + 1) - \ln n = \lim_{n \rightarrow \infty} \ln \left(\frac{n^2 + 1}{n} \right) = \lim_{n \rightarrow \infty} \ln \left(\frac{n^2}{n} \right) = \lim_{n \rightarrow \infty} \ln n = \infty$: diverges

Part 2: Work Out

Directions: Present your solutions in the space provided. Show all your work neatly and concisely and box your final answer. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

16. (12 pts) Evaluate $\int \frac{-2x+4}{(x^2+1)(x-1)} dx$
- key:** $-\frac{1}{2} \ln|x^2+1| - 3 \arctan x + \ln|x-1| + C$
- $$\frac{-2x+4}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$$
- $$\Rightarrow -2x+4 = (Ax+B)(x-1) + C(x^2+1)$$
- $x=1: 2 = 0 + C(2) \therefore C=1$
 $x=0: 4 = B(-1) + 1(1) \Rightarrow 3 = B(-1) \therefore B=-3$
 $x=-1: 6 = (-A-3)(-2) + 1(2) \Rightarrow 4 = (-A-3)(-2) \therefore A=-1$
- Then
$$= \int \left(\frac{-x-3}{x^2+1} + \frac{1}{x-1} \right) dx$$
- $$= -\int \frac{x}{x^2+1} dx - \int \frac{3}{x^2+1} dx + \int \frac{1}{x-1} dx$$
- $$= -\frac{1}{2} \ln|x^2+1| - 3 \arctan x + \ln|x-1| + C$$

17. (12 pts) Evaluate $\int \frac{1}{x^2 \sqrt{x^2+4}} dx$
- key:** $-\frac{\sqrt{x^2+4}}{4x} + C$
- Let $x = 2 \tan \theta \Rightarrow dx = 2 \sec^2 \theta d\theta$
- $$= \int \frac{1}{4 \tan^2 \theta \cdot \sqrt{4 \tan^2 \theta + 4}} \cdot 2 \sec^2 \theta d\theta = \int \frac{1}{4 \tan^2 \theta \cdot 2 \sec \theta} \cdot 2 \sec^2 \theta d\theta = \int \frac{1}{4 \tan^2 \theta} \cdot \sec \theta d\theta$$
- $$= \frac{1}{4} \int \frac{\cos^3 \theta}{\sin^2 \theta} d\theta = \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta$$
- Let $u = \sin \theta \Rightarrow du = \cos \theta d\theta$
- $$= \frac{1}{4} \int \frac{1}{u^2} du = -\frac{1}{4u} + C = -\frac{1}{4 \sin \theta} + C$$
- And since $x = 2 \tan \theta \Rightarrow \tan \theta = \frac{x}{2}$
- $$\therefore = -\frac{\sqrt{x^2+4}}{4x} + C$$

18. (8 pts) Consider the following series
- (a) Determine whether the series converges or diverges, and state the reason.
 (b) If it converges, find its sum. If it diverges, write DIVERGES.
- key:** $\frac{7}{24}$
- $$\sum_{n=1}^{\infty} \frac{3^{n-1} + 1}{3^{2n}}$$
- $$= \sum_{n=1}^{\infty} \frac{3^{n-1}}{3^{2n}} + \sum_{n=1}^{\infty} \frac{1}{3^{2n}} = \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{3^{2n}}$$
- $$= \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{1}{3} \right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{9} \right)^n$$
 : geometric
- Since $|r_1| = \frac{1}{3} < 1$ and $|r_2| = \frac{1}{9} < 1$, the series converges.
- And its sum is
- $$= \frac{1}{3} \cdot \frac{\frac{1}{3}}{1 - \frac{1}{3}} + \frac{\frac{1}{9}}{1 - \frac{1}{9}}$$
- $$= \frac{1}{3} \cdot \frac{\frac{1}{3}}{\frac{2}{3}} + \frac{\frac{1}{9}}{\frac{8}{9}} = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{8} = \frac{1}{6} + \frac{1}{8} = \frac{7}{24}$$

19. (8 pts) Use the Integral Test to determine whether $\sum_{n=2}^{\infty} \frac{1}{x(\ln x)^2}$ converges or diverges. Support your answer.
- key:** $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln 2}$. Since the improper integral converges, so the series converges by the Integral Test.
- Let $f(x) = \frac{1}{x(\ln x)^2}$: continuous, positive, and decreasing in $(2, \infty)$
- Then the integral test:
- $$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx$$
- Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$
- $$= \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} \frac{1}{u^2} du = \lim_{t \rightarrow \infty} \left[-\frac{1}{u} \right]_{\ln 2}^{\ln t} = \lim_{t \rightarrow \infty} \left[-\frac{1}{\ln t} + \frac{1}{\ln 2} \right] = \frac{1}{\ln 2}$$
 : converges
- Since the improper integral converges, so the series converges by the Integral Test.