MATH 152, Fall 2022

COMMON EXAM III - VERSION A

LAST NAME(print): ______________________ FIRST NAME(print): ______________________

INSTRUCTOR: Key

SECTION NUMBER: ____________

DIRECTIONS:

1. The use of a calculator, laptop or computer is prohibited.
2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
3. In Part 1, mark the correct choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*
4. In Part 2, present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.
5. **Be sure to *fill in your name, UIN, section number and version letter of the exam on the ScanTron form.***

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ______________________

PART I: Multiple Choice. 4 points each

1. Suppose that $0 \leq a_n \leq b_n$ for every positive integer n. Which of the following statements is always true?

   (a) If $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

   (b) If $\sum_{n=1}^{\infty} a_n$ is divergent, then so is $\sum_{n=1}^{\infty} b_n$.

   (c) If $\sum_{n=1}^{\infty} a_n$ is convergent, then so is $\sum_{n=1}^{\infty} b_n$.

   (d) none of these are always true.

   (e) If $\sum_{n=1}^{\infty} b_n$ is divergent, then so is $\sum_{n=1}^{\infty} a_n$.

2. Write $f(x) = \dfrac{x^3}{1+4x^2}$ as a power series centered at 0.

   (a) $\sum_{n=0}^{\infty} 4^n x^{2n+3}$

   (b) $\sum_{n=0}^{\infty} (-4)^n x^{2n+6}$

   (c) $\sum_{n=0}^{\infty} 4^n x^{2n+6}$

   (d) $\sum_{n=0}^{\infty} (-4)^n x^{2n+3}$

   (e) $\sum_{n=0}^{\infty} (-4)^n x^{2n}$

$$\frac{x^3}{1+4x^2} = x^3 \cdot \frac{1}{1-(-4x^2)} = x^3 \sum_{n=0} (-4x^2)^n$$

$$= x^3 \sum_{n=0} (-1)^n 4^n x^{2n}$$

$$= \sum_{n=0} (-1)^n 4^n x^{2n+3} = \sum_{n=0} (-4)^n x^{2n+3}$$

3. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \dfrac{4^n(x-3)^n}{n!}$.

   (a) ∞

   (b) 0

   (c) None of these.

   (d) $\dfrac{1}{4}$

   (e) 4

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{4^{n+1}(x-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{4^n(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{4(x-3)}{n+1} \right| = 0$$

4. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{n!(4x-1)^n}{3^n}$.

(a) None of these.

(b) $\frac{1}{4}$

(c) 0

(d) $\frac{3}{4}$

(e) ∞

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,(4x-1)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n!\,(4x-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)(4x-1)}{3}\right| = \infty$$

5. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n\, 5^{2n}}{3^{2n+1}\,(2n)!}$

(a) $\frac{1}{3}\sin\left(\frac{5}{3}\right)$

(b) $\frac{1}{3}\arctan\left(\frac{5}{3}\right)$

(c) None of these

(d) $\frac{1}{3}e^{-5/3}$

(e) $\frac{1}{3}\cos\left(\frac{5}{3}\right)$

$$\cos(x) = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{1}{3}\sum_{n=0} \frac{(-1)^n 5^{2n}}{3^{2n}(2n)!} = \frac{1}{3}\cos\left(\frac{5}{3}\right)$$

6. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges to s. based on the Alternating Series Estimation Theorem, which statement is true?

(a) $|R_7| = |s - s_7| < \frac{1}{7}$

(b) None of these.

(c) $|R_7| = |s - s_7| < \frac{1}{8}$

(d) $|R_7| = |s - s_7| < \frac{1}{49}$

(e) $|R_7| = |s - s_7| < \frac{1}{64}$

$$|R_7| = |s - s_7| < b_8 = \frac{1}{8^2} = \frac{1}{64}$$

7. Find the 15th derivative at $x = 4$ for $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{n3^n}$

(a) $f^{(15)}(4) = \frac{-14!}{3^{15}}$

(b) $f^{(15)}(4) = \frac{14!}{3^{15}}$

(c) $f^{(15)}(4) = \frac{-1}{15(3^{15})}$

(d) $f^{(15)}(4) = \frac{1}{15(3^{15})}$

(e) None of these.

$$c_n = \frac{(-1)^n}{n3^n} = \frac{f^{(n)}(4)}{n!}$$

$$f^{(n)}(4) = \frac{(-1)^n n!}{n3^n}$$

$$f^{(15)}(4) = \frac{-15!}{15 \cdot 3^{15}} = \frac{-14!}{3^{15}}$$

8. Which series is absolutely convergent?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(c) None of these.

(d) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{4/3}}$

(e) $\sum_{n=0}^{\infty} \frac{n!}{2^n}$

$\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ div. harmonic.

$\sum \frac{(-1)^n}{n}$ not Abs. conv.

div. since $p = \frac{1}{2}$

$\sum \left|\frac{(-1)^n}{n^{4/3}}\right| = \sum \frac{1}{n^{4/3}}$ conv. by p-series $p = 4/3 > 1$

Thus $\sum \frac{(-1)^n}{n^{4/3}}$ is Abs conv.

9. Which series is conditionally convergent?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

(b) $\sum_{n=0}^{\infty} \frac{7}{n^5}$

(c) None of these.

(d) $\sum_{n=0}^{\infty} \frac{n!}{2^n}$

(e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$

$\sum \left|\frac{(-1)^n}{n^{3/4}}\right| = \sum \frac{1}{n^{3/4}}$ div by p-series $p = 3/4 < 1$

$\sum \frac{(-1)^n}{n^{3/4}}$ conv. by AST.

10. Use a MacLaurin series to express $f(x) = xe^{2x^2}$ as a power series centered at $x = 0$.

(a) None of these.

(b) $\sum_{n=0}^{\infty} \frac{2^{2n} x^{4n+1}}{(2n)!}$

(c) $\sum_{n=0}^{\infty} \frac{(-2)^n x^{2n+1}}{n!}$

(d) $\sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{n!}$

(e) $\sum_{n=0}^{\infty} \frac{2^n x^{2n}}{n!}$

$e^x = \sum_{n=0} \frac{x^n}{n!}$

$xe^{2x^2} = x \sum_{n=0} \frac{(2x^2)^n}{n!} = \sum_{n=0} \frac{2^n x^{2n+1}}{n!}$

$\sin(x) = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

11. Which of the following is the MacLaurin series for $f(x) = x\sin(x^2)$?

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(2n+1)!}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!}$

$x\sum_{n=0} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = x\sum_{n=0} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$

$= \sum_{n=0} \frac{(-1)^n x^{4n+3}}{(2n+1)!}$

12. Suppose that the series $\sum_{n=1}^{\infty} c_n x^n$ converges at $x = -4$ and diverges at $x = 6$. Which of the following statements is true?

center is $x=0$

(I) $\sum_{n=1}^{\infty} c_n 4^n$ converges. $x=4$

(II) $\sum_{n=1}^{\infty} c_n 7^n$ diverges. $x=7$

(III) $\sum_{n=1}^{\infty} c_n 5^n$ may or may not converge. $x=5$

(a) III only
(b) II and III only
(c) I and II only
(d) I, II, and III
(e) II only

Since conv for $x=-4$ we know that the series will conv for
$-4 \le x < 4$
div. for $x=6$ means the series div. for $x \ge 6$ and $x < -6$
we are unsure about every other value of x.

13. What is the value of the limit, L, that is used in the ratio test for this series? $\sum_{n=1}^{\infty} \frac{n!\, n!\, 3^n}{(2n)!}$.

$(2(n+1))! = (2n+2)!$

(a) $L = \frac{3}{4}$
(b) $L = \infty$
(c) $L = 0$
(d) $L = \frac{3}{2}$
(e) $L = 3$

$\lim_{n\to\infty} \left| \frac{(n+1)!\,(n+1)!\,3^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!\,n!\,3^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)(n+1)\cdot 3}{(2n+2)(2n+1)} \right| = \lim_{n\to\infty} \frac{3n^2+6n+3}{4n^2+6n+2}$

$= \frac{3}{4}$ by L'H

$f(x) = x^{-3}$

14. If we find the Taylor Polynomial for $f(x) = \frac{1}{x^3}$ centered at 7, what is the coefficient of the $(x-7)^3$ term?

(a) $\frac{6!}{7^6}$

(b) $\frac{-10}{7^6}$

(c) $\frac{10}{7^6}$

(d) $\frac{-6!}{7^6}$

(e) $\frac{-10}{x^6}$

$f' = -3x^{-4}$

$f'' = (-3)(-4)x^{-5}$

$f''' = (-3)(-4)(-5)x^{-6}$

$= -\frac{3(4)(5)}{x^6}$

$f'''(7) = -\frac{3(4)(5)}{7^6}$

need $c_3 = \frac{f^{(3)}(7)}{3!}$

$c_3 = \frac{1}{3!} \cdot \frac{-3(4)(5)}{7^6}$

$= \frac{-10}{7^6}$

15. Which of these is the MacLaurin series for $f(x) = x^4 \arctan(2x)$?

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+5}}{2n+1}$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} x^{2n+5}}{(2n+1)!}$

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+5}}{2n+1}$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+5}}{2n+1}$

(e) None of these.

$\arctan(x) = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1}$

$\hookrightarrow x^4 \sum_{n=0} \frac{(-1)^n (2x)^{2n+1}}{2n+1} = x^4 \sum_{n=0} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$

$= \sum_{n=0} \frac{(-1)^n 2^{2n+1} x^{2n+5}}{2n+1}$

16. Find the Taylor polynomial $T_4(x)$, the 4th degree Taylor polynomial, for the function $f(x) = \frac{1}{1+5x^2}$ centered at $a = 0$?

(a) $T_4(x) = 1 + 5x^2 + 25x^4 + 125x^6$

(b) $T_4(x) = 1 + 5x^2 - 25x^4 + 125x^6$

(c) $T_4(x) = 1 - 5x^2 + 25x^4$

(d) $T_4(x) = 1 + 5x^2 + 25x^4$

(e) $T_4(x) = 1 - 5x^2 + 25x^4 - 125x^6$

$\frac{1}{1 - -5x^2} = \sum_{n=0} (-5x^2)^n$

$= \sum_{n=0} (-1)^n 5^n x^{2n}$

$= \underbrace{1 - 5x^2 + 25x^4}_{T_4} - 125x^6$

T_4 has degree 4

PART II WORK OUT

Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *Box your final answer.* You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

17. (5 points each) Determine whether the following series converge or diverge. Clearly explain your reasoning and state any tests used.

(a) $\sum_{n=1}^{\infty} \frac{5+2\sin n}{n}$

$$\frac{3}{n} \leq \frac{5+2\sin(n)}{n} \leq \frac{7}{n}$$

$\sum_{n=1} \frac{3}{n}$ div. by p-series $(p=1)$

by the comparison thrm. we get $\sum_{n\geq 1} \frac{5+2\sin(n)}{n}$ will diverge

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+2}$

$b_n = \frac{1}{n^4+2}$ $\qquad \lim_{n\to\infty} b_n = 0$

$$f(x) = \frac{1}{x^4+2} = (x^4+2)^{-1}$$

$$f' = -(x^4+2)^{-2}(4x^3)$$

$f' = \frac{-4x^3}{(x^4+2)^2} < 0$ for $x \geq 1$ $\quad$ so $f(x)$ is dec. and thus b_n are dec.

by AST the series converges

18. (8 points) Find the MacLaurin series representation for the function $f(x) = \dfrac{1}{(1-7x)^2}$.

$$g(x) = \frac{1}{1-7x} = \sum_{n=0} 7^n x^n$$

$$g(x) = (1-7x)^{-1}$$

$$g'(x) = -(1-7x)^{-2}(-7)$$

$$= \frac{7}{(1-7x)^2} = \sum_{n=1} 7^n n x^{n-1}$$

$$f(x) = \frac{1}{(1-7x)^2} = \frac{1}{7} g'(x) = \frac{1}{7} \sum_{n=1} 7^n n x^{n-1}$$

$$f(x) = \sum_{n=1} 7^{n-1} n x^{n-1}$$

If shifting the index to start at zero is required

Then $$f(x) = \sum_{n=0} 7^n (n+1) x^n$$

19. (10 points) Find the radius of convergence and the interval of convergence of the power series. You must test your endpoints for convergence.

$$\sum_{n=1}^{\infty} \frac{(-2)^n (x+4)^n}{n6^n}$$

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}(x+4)^{n}}{(n+1)6^{n+1}} \cdot \frac{n6^n}{(-2)^n (x+4)^n} \right| = \lim_{n\to\infty} \left| \frac{-2(x+4)n}{(n+1)6} \right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \frac{|x+4)}{3} = \frac{|x+4|}{3} < 1$$

$|x+4| < 3$ $R = 3$

$-3 < x+4 < 3$

$-7 < x < -1$ $I: (-7,-1]$ OR $-7 < x \leq -1$

Test end points

$x = -1$

$$\sum_{n=1} \frac{(-2)^n (3)^n}{n6^n} = \sum_{n=1} \frac{(-1)^n 2^n 3^n}{n6^n} = \sum_{n=1} \frac{(-1)^n}{n}$$

converge by AST $b_n = \frac{1}{n}$

$x = -7$

$$\sum_{n=1} \frac{(-2)^n(-3)^n}{n6^n} = \sum_{n=1} \frac{2^n 3^n}{n6^n} = \sum_{n=1} \frac{1}{n}$$

diverge by p-series $p = 1$

20. (8 points) Find the Taylor series for $f(x) = xe^x$ about $a = 4$. Express your answer in summation notation.

$f(x) = xe^x$

$f' = e^x + xe^x = (1+x)e^x$

$f'' = 1e^x + (1+x)e^x$

$= (2+x)e^x$

$f''' = 1e^x + (2+x)e^x$

$= (3+x)e^x$

$f^{(4)} = 1e^x + (3+x)e^x$

$= (4+x)e^x$

$f^{(n)}(x) = (n+x)e^x$

$C_n = \dfrac{f^n(4)}{n!} = \dfrac{(n+4)e^4}{n!}$

$$f(x) = \sum_{n=0} C_n (x-4)^n = \sum_{n=0} \frac{(n+4)e^4}{n!}(x-4)^n$$

DO NOT WRITE IN THIS TABLE.

Question	Points Awarded	Points
1-16		64
17		10
18		8
19		10
20		8
		100