MATH 152, Spring 2022

EXAM III - VERSION A

LAST NAME(print): solutions ______ FIRST NAME(print): ______

UIN: ______

INSTRUCTOR: ______

SECTION NUMBER: ______

DIRECTIONS:

1. No calculator, cell phones, or other electronic devices may be used, and they must all be put away out of sight.
2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
3. In Part 1, mark the choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*
4. In Part 2, present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer.* You will be graded not merely on the final answer, but also on the quality and ness of the work leading up to it.
5. **Be sure to *fill in your name, UIN, section number and version letter of the exam* on the ScanTron *form.***

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ______

PART I: Multiple Choice. 4 points each.

1. Which of the following statements is true for the three series given below?

(I) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^3}$ (II) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ (III) $\sum_{n=2}^{\infty} \frac{(-1)^n (n)}{\ln n}$

(a) I converges absolutely, II and III converge conditionally.

(b) I converges absolutely, II converges conditionally, and III diverges .

(c) I and II converge conditionally, and III diverges.

(d) I converges conditionally, II and III diverge.

(e) I and II converge absolutely and III converges conditionally.

I. CA since $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges by I.T.

II. CC since $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ converges by AST but $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges by I.T.

III Diverges by T.D. $\lim_{n\to\infty} \frac{(-1)^n n}{\ln n} \neq 0$

2. What is the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-2)^n(2n+1)!}{(n+1)!}$?

(a) $\{2\}$

(b) $(-\infty, \infty)$

(c) $[1.5, 2.5]$

(d) $\{0\}$

(e) $[1.5, 2.5)$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}(2n+3)!}{(n+2)!} \cdot \frac{(n+1)!}{(x-2)^n(2n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(x-2)(2n+3)(2n+2)}{n+2} \right| = \infty$$ unless $x = 2$.

$I = \{2\}$

3. If we find the third degree Taylor Polynomial for $f(x) = e^{-2x}$ centered at 4, what is the coefficient of $(x-4)^3$?

(a) $-\frac{8}{3}e^{-8}$

(b) $-\frac{4}{3}e^{-8}$

(c) $\frac{2}{3}e^{-8}$

(d) $\frac{4}{3}e^{-8}$

(e) $\frac{8}{3}e^{-8}$

$T_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$

coefficient of $(x-4)^3$ is therefore $\boxed{\frac{f'''(4)}{3!}}$.

$f = e^{-2x}$

$f' = -2e^{-2x}$

$f'' = 4e^{-2x}$

$f''' = -8e^{-2x}$

$f'''(x) = -8e^{-2x}$

so $f'''(4) = -8e^{-8}$

$\frac{f'''(4)}{3!} = \frac{-8e^{-8}}{6} = \frac{-4e^{-8}}{3}$

4. If $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{5^n}(x-2)^n$, find $f^{(25)}(2)$, that is, the 25^{th} derivative of $f(x)$ evaluated at $x = 2$.

(a) $f^{(25)}(2) = \frac{3^{(26)}(25)!}{5^{(25)}}$

(b) $f^{(25)}(2) = \frac{(-1)3^{(25)}}{5^{(25)}(25)!}$

(c) $f^{(25)}(2) = \frac{(-1)3^{(26)}}{5^{(26)}(25)!}$

(d) $f^{(25)}(2) = \frac{3^{(25)}(25)!}{5^{(26)}}$

(e) $f^{(25)}(2) = \frac{(-1)3^{(26)}(25)!}{5^{(25)}}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(2)}{n!}(x-2)^n,$$

equate coefficients of $(x-2)^n$

$$\frac{f^{n}(2)}{n!} = \frac{(-1)^n 3^{n+1}}{5^n}$$

let $n = 25$, cross multiply by $n!$

$$f^{25}(2) = \frac{(25)!(-1)(3^{26})}{5^{25}}$$

5. When we apply the Ratio Test to the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{2n}}{n^2+100}$, we find

(a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$

(b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$

(c) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{9}$

(d) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3$

(e) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 9$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 9^n}{n^2+100}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2} 9^{n+1}}{(n+1)^2+100} \cdot \frac{n^2+100}{(-1)^{n+1} 9^n} \right| = 9$$

6. Which of the following is a power series representation for $f(x) = \frac{x}{x^3+8}$?

(a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{8^{n+1}}$, $|x| < 2$

(b) $f(x) = \sum_{n=0}^{\infty} \frac{x^{3n+3}}{8^{n+1}}$, $|x| < 2$

(c) $f(x) = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{8^{n+1}}$, $|x| < \frac{1}{2}$

(d) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{8^{n+1}}$, $|x| < \frac{1}{2}$

(e) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{8^{n+1}}$, $|x| < 2$

$$= \frac{x}{8(1+\frac{x^3}{8})}$$

$$= \frac{x}{8}\sum_{n=0}^{\infty}\left(-\frac{x^3}{8}\right)^n, \qquad \left|-\frac{x^3}{8}\right| < 1$$

$$|x| < 2$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{8^{n+1}}$$

7. Which of the following is a power series representation for $f(x) = \dfrac{1}{(1-3x)^2}$?

(a) $f(x) = \sum_{n=0}^{\infty} 3^n n x^{n-1}$, $|x| < \frac{1}{3}$

(b) $f(x) = -\sum_{n=0}^{\infty} 3^{n-1}(n+1)x^n$, $|x| < \frac{1}{3}$

(c) $f(x) = \sum_{n=0}^{\infty} 3^n (n+1)x^n$, $|x| < \frac{1}{3}$

(d) $f(x) = -\sum_{n=0}^{\infty} 3^n n x^{n-1}$, $|x| < \frac{1}{3}$

(e) $f(x) = \sum_{n=0}^{\infty} 3^{n-1}(n+1)x^n$, $|x| < \frac{1}{3}$

$$\int \frac{dx}{(1-3x)^2} = \frac{1}{3}\cdot\frac{1}{1-3x} = \frac{1}{3}\sum_{n=0}^{\infty}(3x)^n$$

$$\frac{1}{(1-3x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty} 3^{n-1}x^n, \quad |3x|<1, \quad |x|<\frac{1}{3}$$

$$= \sum_{n=1}^{\infty} 3^{n-1} n x^{n-1}$$

$$= \sum_{n=0}^{\infty} 3^n (n+1) x^n$$

8. Find the sum of the series $\sum_{n=0}^{\infty} \dfrac{(-1)^n 3^n (\pi)^{2n}}{n!}$.

(a) 0

(b) -1

(c) $\cos(3\pi^2)$

(d) $e^{-3\pi^2}$

(e) $e^{3\pi^2}$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n 3^n (\pi^2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-3\pi^2)^n}{n!}$$

$$= e^{-3\pi^2}$$

1.

9. Using The Alternating Series Estimation Theorem, what is the minimum number of terms needed to find the sum of the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n^3}$ to within $\dfrac{1}{165}$?

(a) $n = 6$

(b) $n = 5$

(c) $n = 4$

(d) $n = 3$

(e) $n = 7$

$|R_n| \leq b_{n+1} = \dfrac{1}{(n+1)^3}$, and the smallest value of n that makes

$\dfrac{1}{(n+1)^3} < \dfrac{1}{165}$ is $n=5$, since

$\dfrac{1}{6^3} = \dfrac{1}{216}$ yet

$\dfrac{1}{5^3} = \dfrac{1}{125}$

The R.T. is inconclusive if the series is void of factorials and exponentials, which is true for (b)

10. For which of the following series is the Ratio Test inconclusive?

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n!}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n}$

(d) $\sum_{n=1}^{\infty} \frac{n! \sin(n)}{8^n}$

(e) $\sum_{n=1}^{\infty} \frac{\cos(n)}{n!}$

(b) only since this is the only series that yields $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$

note: for (a), and (e) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.$

for (d), $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty.$

for (c), $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{4}$

11. Given the series $\sum_{n=1}^{\infty} c_n (x+2)^n$ converges at $x = 3$. Which of the following statements **must** be true?

(a) The series $\sum_{n=1}^{\infty} c_n (x+2)^n$ diverges at $x = 4$.

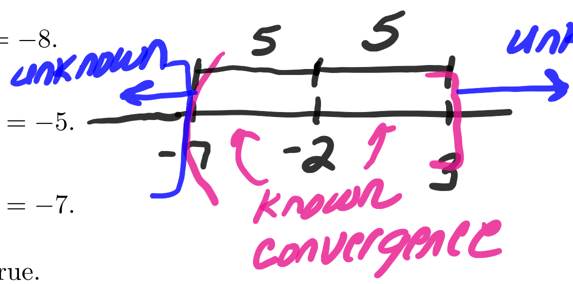
(b) The series $\sum_{n=1}^{\infty} c_n (x+2)^n$ diverges at $x = -8$.

(c) The series $\sum_{n=1}^{\infty} c_n (x+2)^n$ converges at $x = -5$.

(d) The series $\sum_{n=1}^{\infty} c_n (x+2)^n$ converges at $x = -7$.

(e) None of the above statements must be true.

center of series is -2.



12. Which of the following is the correct Maclaurin series for $f(x) = x \sin(5x)$?

(a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+1}}{(2n+1)!}$

(b) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+2} x^{2n+1}}{(2n+1)!}$

(c) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+2}}{(2n+1)!}$

(d) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{2n+2}}{(2n+1)!}$

(e) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+2} x^{2n+2}}{(2n+1)!}$

$$f(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{2n+1}}{(2n+1)!}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+2}}{(2n+1)!}$$

13. The series $\sum_{n=1}^{\infty} \frac{\sin(n)+5}{n^{3/2}}$

(a) Converges by the comparison test with $\sum_{n=1}^{\infty} \frac{4}{n^{3/2}}$

(b) Converges by the comparison test with $\sum_{n=1}^{\infty} \frac{5}{n^{3/2}}$

(c) Converges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

(d) Converges by the comparison test with $\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}$

(e) None of the above

$\frac{\sin(n)+5}{n^{3/2}} \leq \frac{6}{n^{3/2}}$

and $\Sigma \frac{6}{n^{3/2}}$ converges by p-series

so both converge by comparison test.

14. For what values of p does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge?

(a) Only if $p > 1$.

(b) Only if $p \geq 0$.

(c) Only if $p \geq 1$.

(d) Only if $p > 0$.

(e) The series converges for all p.

By AST, if $p>0$, $b_n = \frac{1}{n^p}$ decreases to zero

True or False. On your scantron, bubble 'a' if true and bubble 'b' if false. One point each.

15. If $0 \leq a_n \leq b_n$ for every positive integer n and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges. False

16. If $0 \leq a_n \leq b_n$ for every positive integer n and $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. True

17. If $0 \leq a_n \leq b_n$ for every positive integer n and $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} b_n$ diverges. True

18. If $0 \leq a_n \leq b_n$ for every positive integer n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ converges. False

PART II: Free Response: Show all work and box your final answer!

19. (12 pts) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x+3)^n}{4^n(2n+1)}$. Be sure to test the endpoints of the interval for convergence.

RT: $\lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{4^{n+1}(2n+3)} \cdot \frac{4^n(2n+1)}{(x+3)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x+3)(2n+1)}{4(2n+3)} \right| = \left| \frac{x+3}{4} \right|$.

$\frac{|x+3|}{4} < 1 \rightarrow |x+3| < 4 \rightarrow -4 < x+3 < 4$

$-7 < x < 1$

$\boxed{R=4}$

Test endpoints: • $x=-7$ $\sum_{n=1}^{\infty} \frac{(-4)^n}{4^n(2n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

AST: $b_n = \frac{1}{2n+1}$ $\lim_{n\to\infty} \frac{1}{2n+1} = 0$

$\frac{1}{2n+3} < \frac{1}{2n+1} \rightarrow b_n$ decreases

so by AST, the series converges at $x=-7$

• $x=1$: $\sum_{n=1}^{\infty} \frac{4^n}{4^n(2n+1)} = \sum_{n=1}^{\infty} \frac{1}{2n+1}$

which diverges by either integral test: $\int_1^{\infty} \frac{dx}{2x+1}$

$\frac{1}{2} \ln|2x+1| \Big|_1^{\infty} = \infty$

or LCT $b_n = \frac{1}{2n}$ $\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{2n}} = 1 > 0$ and $\sum \frac{1}{2n}$ diverge, so both diverge

series diverges at $x=1$ $I = [-7, 1)$

20. Consider $f(x) = x^3 \cos\left(\frac{x}{3}\right)$.

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

a.) (5 pts) Using the known Maclaurin series for $\cos x$, write $f(x) = x^3 \cos\left(\frac{x}{3}\right)$ as a Maclaurin series. Include the radius of convergence.

$$x^3 \cos\left(\frac{x}{3}\right) = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{3}\right)^{2n}}{(2n)!} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{3^{2n}(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{3^{2n}(2n)!}, \quad R = \infty$$

b.) (4 pts) Using the result above, evaluate $\int_0^{0.1} x^3 \cos\left(\frac{x}{3}\right) dx$ as a series.

$$\int_0^{0.1} x^3 \cos\left(\frac{x}{3}\right) dx = \int_0^{0.1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{3^{2n}(2n)!} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{3^{2n}(2n)!(2n+4)} \Bigg|_0^{0.1} = \sum_{n=0}^{\infty} \frac{(-1)^n (0.1)^{2n+4}}{3^{2n}(2n)!(2n+4)}$$

21. **Using a comparison test or limit comparison test**, determine whether the series below converge or diverge. Show all work, as illustrated in class, by naming the test, applying the test, and drawing the correct conclusion.

(a) (4 pts) $\sum_{n=1}^{\infty} \frac{2^n}{n+5^n}$ $\leq \sum_{n=1}^{\infty} \frac{2^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$,

a convergent geometric series

$|r| = \left|\frac{2}{5}\right| < 1$. By CT, both series converge

(b) (4 pts) $\sum_{n=3}^{\infty} \frac{1}{\sqrt[3]{4n^3+4n+1}}$ CT fails. LCT $b_n = \frac{1}{\sqrt[3]{4n^3}}$

or $b_n = \frac{1}{n}$, which will still give a positive finite limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\frac{1}{\sqrt[3]{4n^3+4n+1}}}{\frac{1}{\sqrt[3]{4n^3}}} \right) = 1 > 0$$

and $\sum_{n=3}^{\infty} \frac{1}{\sqrt[3]{4n^3}}$ is a divergent p-series $p=1$. Both diverge

note: this is not a series of positive terms, so we must test for absolute convergence

(c) (4 pts) $\sum_{n=1}^{\infty} \frac{\cos n}{n^4+n^2+\sqrt{n}}$

$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^4+n^2+\sqrt{n}} \right| < \sum_{n=1}^{\infty} \frac{1}{n^4}$, a convergent p-series, $p=4$

original series converges absolutely thus converges.

22. (7 pts) Find the Taylor Series for $f(x) = \dfrac{1}{x^3}$ centered at 5. You do not need to find the radius or interval of convergence.

$$f(x) = \frac{1}{x^3} = \sum_{n=0}^{\infty} \frac{f^n(5)}{n!}(x-5)^n$$

$$f(x) = \frac{1}{x^3}$$

$$f'(x) = \frac{-3}{x^4}$$

$$f''(x) = \frac{4\cdot 3}{x^5} = \frac{4!}{2x^5}$$

$$f'''(x) = \frac{-5\cdot 4\cdot 3}{x^6} = -\frac{5!}{2x^6}$$

$$f^n(x) = \frac{(-1)^n(n+2)!}{2x^{n+3}}$$

$$f^n(5) = \frac{(-1)^n(n+2)!}{(2)5^{n+3}}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)!}{(2)5^{n+3}\,n!}(x-5)^n$$

or, since $(n+2)! = (n+2)(n+1)n!$,

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(n+2)(n+1)(x-5)^n}{(2)5^{n+3}}$$

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Question	Points Awarded	Points
1-18		60
19		12
20		9
21		12
22		7
		100

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