MATH 152, Spring 2022

EXAM III - VERSION B

LAST NAME(print): Solutions ______ FIRST NAME(print): ______

UIN: ______

INSTRUCTOR: ______

SECTION NUMBER: ______

DIRECTIONS:

1. No calculator, cell phones, or other electronic devices may be used, and they must all be put away out of sight.
2. TURN OFF cell phones and put them away. If a cell phone is seen during the exam, your exam will be collected and you will receive a zero.
3. In Part 1, mark the choice on your ScanTron using a No. 2 pencil. The scantrons will not be returned, therefore *for your own records, also record your choices on your exam!*
4. In Part 2, present your solutions in the space provided. *Show all your work* neatly and concisely and *clearly indicate your final answer*. You will be graded not merely on the final answer, but also on the quality and ness of the work leading up to it.
5. **Be sure to *fill in your name, UIN, section number and version letter of the exam* on the ScanTron *form.***

THE AGGIE CODE OF HONOR

"An Aggie does not lie, cheat or steal, or tolerate those who do."

Signature: ______

PART I: Multiple Choice. 4 points each.

1. For which of the following series is the Ratio Test inconclusive?

   (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n!}$

   (b) $\sum_{n=1}^{\infty} \frac{\cos(n)}{n!}$

   (c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n}$

   (d) $\sum_{n=1}^{\infty} \frac{n!\sin(n)}{8^n}$

   (e) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n}$

If the series is void of factorials / exponentials, RT will fail. (e) is only series where $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$.

For (a) and (b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$

For (d) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$ for (c) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$

2. For what values of p does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge?

   (a) Only if $p > 1$.

   (b) Only if $p > 0$.

   (c) Only if $p \geq 1$.

   (d) Only if $p \geq 0$.

   (e) The series converges for all p.

If $p>0$, $b_n = \frac{1}{n^p}$ which decreases to 0.

3. What is the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-2)^n(2n+1)!}{(n+1)!}$?

   (a) $\{0\}$

   (b) $(-\infty, \infty)$

   (c) $[1.5, 2.5]$

   (d) $\{2\}$

   (e) $[1.5, 2.5)$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}(2n+3)!}{(n+2)!} \cdot \frac{(n+1)!}{(x-2)^n(2n+1)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-2)(2n+3)(2n+1)}{n+2} \right| = \infty \text{ unless } x=2$$

$I = \{2\}$

4. Given the series $\sum_{n=1}^{\infty} c_n(x+2)^n$ converges at $x=3$. Which of the following statements **must** be true?

series is centered at $x=-2$

(a) The series $\sum_{n=1}^{\infty} c_n(x+2)^n$ diverges at $x=4$.

(b) The series $\sum_{n=1}^{\infty} c_n(x+2)^n$ converges at $x=-5$.

(c) The series $\sum_{n=1}^{\infty} c_n(x+2)^n$ diverges at $x=-8$.

(d) The series $\sum_{n=1}^{\infty} c_n(x+2)^n$ converges at $x=-7$.

(e) None of the above statements must be true.

unknown conv unknown

-7 -2 3

5. Which of the following statements is true for the three series given below?

(I) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^3}$ (II) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{n}$ (III) $\sum_{n=2}^{\infty} \frac{(-1)^n (n)}{\ln n}$

(a) I converges absolutely, II and III converge conditionally.

(b) I and II converge conditionally, and III diverges.

(c) I converges absolutely, II converges conditionally, and III diverges .

(d) I converges conditionally, II and III diverge.

(e) I and II converge absolutely and III converges conditionally.

I CA since $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges by I.T

II CC since $\sum \frac{(-1)^n \ln n}{n}$ converges by AST but $\sum \frac{\ln n}{n}$ diverges by I.T

III diverges by T.D.

6. Which of the following is the correct Maclaurin series for $f(x) = x\sin(5x)$?

(a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+2}}{(2n+1)!}$

(b) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+2} x^{2n+1}}{(2n+1)!}$

(c) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+1}}{(2n+1)!}$

(d) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^n x^{2n+2}}{(2n+1)!}$

(e) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+2} x^{2n+2}}{(2n+1)!}$

$$x \sum_{n=0}^{\infty} \frac{(-1)^n (5x)^{2n+1}}{(2n+1)!}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1} x^{2n+2}}{(2n+1)!}$$

7. The series $\sum_{n=1}^{\infty} \frac{\sin(n)+5}{n^{3/2}}$ $\le \sum_{n=1}^{\infty} \frac{6}{n^{3/2}}$, a convergent p-series

(a) Converges by the comparison test with $\sum_{n=1}^{\infty} \frac{4}{n^{3/2}}$

(b) Converges by the comparison test with $\sum_{n=1}^{\infty} \frac{5}{n^{3/2}}$

(c) Converges by the comparison test with $\sum_{n=1}^{\infty} \frac{6}{n^{3/2}}$

(d) Converges by the comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

(e) None of the above

8. If $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{5^n}(x-2)^n$, find $f^{(25)}(2)$, that is, the 25^{th} derivative of $f(x)$ evaluated at $x = 2$.

(a) $f^{(25)}(2) = \frac{(-1)3^{(26)}(25)!}{5^{(25)}}$

(b) $f^{(25)}(2) = \frac{(-1)3^{(25)}}{5^{(25)}(25)!}$

(c) $f^{(25)}(2) = \frac{(-1)3^{(26)}}{5^{(26)}(25)!}$

(d) $f^{(25)}(2) = \frac{3^{(25)}(25)!}{5^{(26)}}$

(e) $f^{(25)}(2) = \frac{3^{(26)}(25)!}{5^{(25)}}$

$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(2)}{n!}(x-2)^n$ equate coefficients

so $\frac{f^{n}(2)}{n!} = \frac{(-1)^n 3^{n+1}}{5^n}$, so let $n=25$

$f^{25}(2) = \frac{(25)!(-1)3^{26}}{5^{25}}$

9. If we find the third degree Taylor Polynomial for $f(x) = e^{-2x}$ centered at 4, what is the coefficient of $(x-4)^3$?

(a) $-\frac{8}{3}e^{-8}$

(b) $\frac{8}{3}e^{-8}$

(c) $\frac{2}{3}e^{-8}$

(d) $\frac{4}{3}e^{-8}$

(e) $-\frac{4}{3}e^{-8}$

$T_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3.$

The coefficient of $(x-4)^3$ is $\frac{f'''(4)}{3!}$.

$f'''(4) = -8e^{-2x}$, so $f'''(4) = 8e^{-8}$

$\frac{f'''(4)}{3!} = \frac{-8e^{-8}}{6} = -\frac{4}{3}e^{-8}$

10. Which of the following is a power series representation for $f(x) = \dfrac{1}{(1-3x)^2}$?

(a) $f(x) = \sum_{n=0}^{\infty} 3^n n x^{n-1}$, $|x| < \frac{1}{3}$

(b) $f(x) = \sum_{n=0}^{\infty} 3^n (n+1) x^n$, $|x| < \frac{1}{3}$

(c) $f(x) = -\sum_{n=0}^{\infty} 3^{n-1} (n+1) x^n$, $|x| < \frac{1}{3}$

(d) $f(x) = -\sum_{n=0}^{\infty} 3^n n x^{n-1}$, $|x| < \frac{1}{3}$

(e) $f(x) = \sum_{n=0}^{\infty} 3^{n-1} (n+1) x^n$, $|x| < \frac{1}{3}$

$$\int \frac{dx}{(1-3x)^2} = \frac{1}{3}\cdot\frac{1}{1-3x} = \frac{1}{3}\sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty} 3^{n-1}x^n$$

$$\frac{1}{1-3x} = \frac{d}{dx}\sum_{n=0}^{\infty} 3^{n-1}x^n = \sum_{n=1}^{\infty} 3^{n-1} n x^{n-1} = \sum_{n=0}^{\infty} 3^n(n+1)x^n$$

$$|x| < \frac{1}{3}$$

11. Find the sum of the series $\sum_{n=0}^{\infty} \dfrac{(-1)^n 3^n (\pi)^{2n}}{n!}$.

(a) 0

(b) -1

(c) $\cos(3\pi^2)$

(d) $e^{3\pi^2}$

(e) $e^{-3\pi^2}$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^n (\pi^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3\pi^2)^n}{n!}$$

$$= e^{-3\pi^2}$$

12. Which of the following is a power series representation for $f(x) = \dfrac{x}{x^3+8}$?

(a) $f(x) = \sum_{n=0}^{\infty} \dfrac{x^{3n+1}}{8^{n+1}}$, $|x| < \frac{1}{2}$

(b) $f(x) = \sum_{n=0}^{\infty} \dfrac{x^{3n+3}}{8^{n+1}}$, $|x| < 2$

(c) $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n+1}}{8^{n+1}}$, $|x| < 2$

(d) $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n+1}}{8^{n+1}}$, $|x| < \frac{1}{2}$

(e) $f(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{3n+3}}{8^{n+1}}$, $|x| < 2$

$$\frac{x}{8\left(1+\frac{x^3}{8}\right)} = \frac{x}{8}\sum_{n=0}^{\infty}\left(-\frac{x^3}{8}\right)^n,$$

$$\left|-\frac{x^3}{8}\right| < 1$$

$$|x| < 2$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+1}}{8^{n+1}}$$

13. When we apply the Ratio Test to the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{2n}}{n^2+100}$, we find

(a) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$

(b) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$

(c) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{9}$

(d) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3$

(e) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 9$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 9^n}{n^2+100}$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2} 9^{n+1}}{(n+1)^2+100} \cdot \frac{n^2+100}{(-1)^{n+1} 9^n} \right| = 9$$

14. Using The Alternating Series Estimation Theorem, what is the minimum number of terms needed to find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ to within $\frac{1}{165}$?

(a) $n = 6$

(b) $n = 3$

(c) $n = 4$

(d) $n = 5$

(e) $n = 7$

$|R_n| \leq b_{n+1} = \frac{1}{(n+1)^3}$.

The smallest value of n that satisfies $\frac{1}{(n+1)^3} < \frac{1}{165}$ is $n=5$, since $\frac{1}{6^3} = \frac{1}{216}$ and $\frac{1}{5^3} = \frac{1}{125}$

True or False. On your scantron, bubble 'a' if true and bubble 'b' if false. One point each.

15. If $0 \leq a_n \leq b_n$ for every positive integer n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ converges. Fase

16. If $0 \leq a_n \leq b_n$ for every positive integer n and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ also converges. False

17. If $0 \leq a_n \leq b_n$ for every positive integer n and $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges. True

18. If $0 \leq a_n \leq b_n$ for every positive integer n and $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} b_n$ diverges. True

PART II: Free Response: Show all work and box your final answer!

19. (12 pts) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x+5)^n}{6^n(4n+1)}$. Be sure to test the endpoints of the interval for convergence.

RT $\lim_{n\to\infty} \left| \frac{(x+5)^{n+1}}{6^{n+1}(4n+5)} \cdot \frac{6^n(4n+1)}{(x+5)^n} \right| = \lim_{n\to\infty} \left| \frac{(x+5)(4n+1)}{6(4n+5)} \right|$

$= \left| \frac{x+5}{6} \right|$. Now, $\left| \frac{x+5}{6} \right| < 1 \to |x+5| < 6$ $\boxed{R=6}$

$-6 < x+5 < 6 \to -11 < x < 1$

Test endpoints: • $x=-11$: $\sum_{n=1}^{\infty} \frac{(-6)^n}{6^n(4n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{6^n}}{\cancel{6^n}(4n+1)}$

AST $b_n = \frac{1}{4n+1}$ $\lim_{n\to\infty} \frac{1}{4n+1} = 0$

convergence at $x=-11$ $\frac{1}{4n+5} < \frac{1}{4n+1} \to b_n$ decreases

• $x=1$: $\sum_{n=1}^{\infty} \frac{6^n}{6^n(4n+5)}$ diverges by LCT or IT

IT: $\int_2^{\infty} \frac{dx}{4x+5} = \frac{1}{4} \ln|4x+5| \Big|_2^{\infty} = \infty$

LCT $\lim_{n\to\infty} \frac{\frac{1}{4n+1}}{\frac{1}{4n}} = 1 > 0$ both diverge since $\sum \frac{1}{4n}$ diverges by p-series $p=1$

series diverges at $x=1$, so $\boxed{I = [-11, 1)}$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

20. Consider $f(x) = x^5 \cos\left(\frac{x}{5}\right)$.

a.) (5 pts) Using the known Maclaurin series for $\cos x$, write $f(x) = x^5 \cos\left(\frac{x}{5}\right)$ as a Maclaurin series. Include the radius of convergence.

$$x^5 \cos\left(\frac{x}{5}\right) = x^5 \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{5}\right)^{2n}}{(2n)!}, \quad R = \infty$$

$$= x^5 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{5^{2n}(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+5}}{5^{2n}(2n)!}$$

$$\text{or} \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+5}}{25^n (2n)!}$$

b.) (4 pts) Using the result above, evaluate $\int_0^{0.2} x^5 \cos\left(\frac{x}{5}\right) dx$ as a series.

$$\int_0^{0.2} x^5 \cos\left(\frac{x}{5}\right) dx = \int_0^{0.2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+5}}{5^{2n}(2n)!} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+6}}{5^{2n}(2n)!(2n+6)} \Bigg|_0^{0.2} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (0.2)^{2n+6}}{5^{2n}(2n)!(2n+6)}}$$

21. **Using a comparison test or limit comparison test**, determine whether the series below converge or diverge. Show all work, as illustrated in class, by naming the test, applying the test, and drawing the correct conclusion.

(a) (4 pts) $\sum_{n=1}^{\infty} \frac{6^n}{n+7^n}$

$< \sum_{n=1}^{\infty} \frac{6^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{6}{7}\right)^n$, a convergent geometric series, $r = \frac{6}{7}$

By CT, both series converge

(b) (4 pts) $\sum_{n=3}^{\infty} \frac{1}{\sqrt[5]{2n^5+3n+1}}$

CT Fails. Try LCT

$b_n = \frac{1}{\sqrt[5]{2n^5}}$ (b_n can be any multiple of $\frac{1}{n}$)

$$\lim_{n\to\infty} \left(\frac{\frac{1}{\sqrt[5]{2n^5+3n+1}}}{\frac{1}{\sqrt[5]{2n^5}}} \right) = 1,$$ and $\sum \frac{1}{\sqrt[5]{2n^5}}$ is a divergent p-series $p=1$.

Both series diverge.

(c) (4 pts) $\sum_{n=1}^{\infty} \frac{\sin n}{n^6+n^4+\sqrt{n}}$

$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^6+n^4+\sqrt{n}} \right| < \sum_{n=1}^{\infty} \frac{1}{n^6}$, a convergent p-series $p=6>1$

so original series converges absolutely, thus converges.

22. (7 pts) Find the Taylor Series for $f(x) = \frac{1}{x^3}$ centered at 3. You do not need to find the radius or interval of convergence.

$$f(x) = \frac{1}{x^3} = \sum_{n=0}^{\infty} \frac{f^{n}(3)}{n!}(x-3)^n$$

$$f(x) = \frac{1}{x^3}$$

$$f'(x) = \frac{-3}{x^4}$$

$$f''(x) = \frac{4 \cdot 3}{x^5} = \frac{4!}{2x^5}$$

$$f'''(x) = \frac{-5 \cdot 4 \cdot 3}{x^6} = -\frac{5!}{2x^6}$$

$$f^{n}(x) = \frac{(-1)^n (n+2)!}{2x^{n+3}}$$

$$f^{n}(3) = \frac{(-1)^n (n+2)!}{(2)3^{n+3}}$$

$$\frac{1}{x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!}{(2)3^{n+3} n!}(x-3)^n$$ or

$$\sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)}{(2) 3^{n+3}}(x-3)^n$$

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Question	Points Awarded	Points
1-18		60
19		12
20		9
21		12
22		7
		100

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